The Wasserstein gradient flow of the Sinkhorn divergence between Gaussian distributions

Mathis Hardion

Joint work with Théo Lacombe (Supervisor)

PhD seminar of the CMAP, May 13, 2026

Introduction

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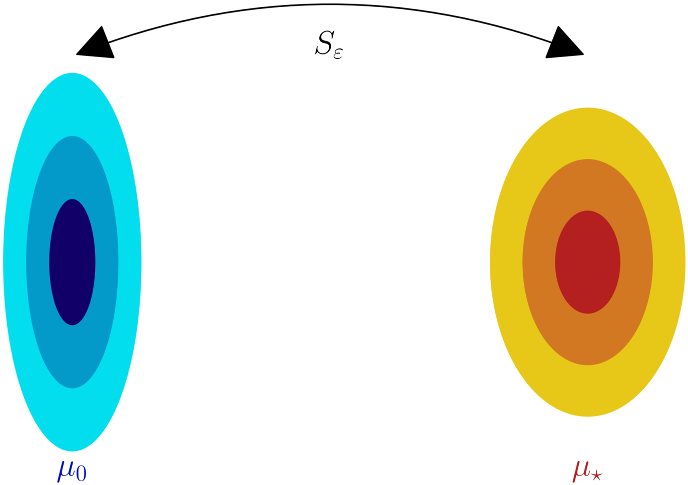

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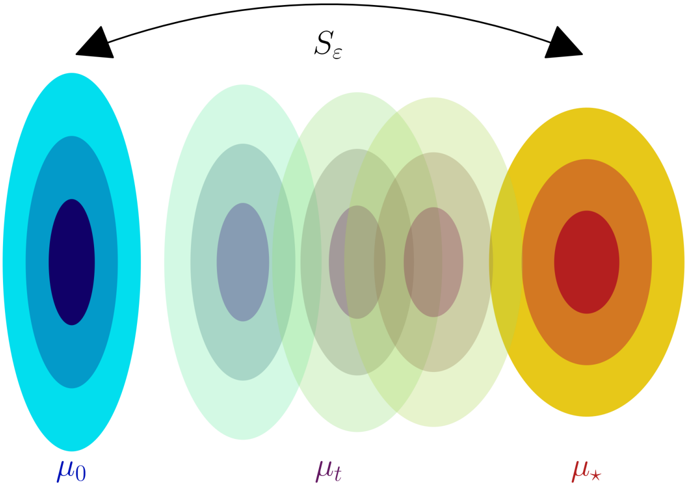

$(\mu_t)_t$ described by a PDE.

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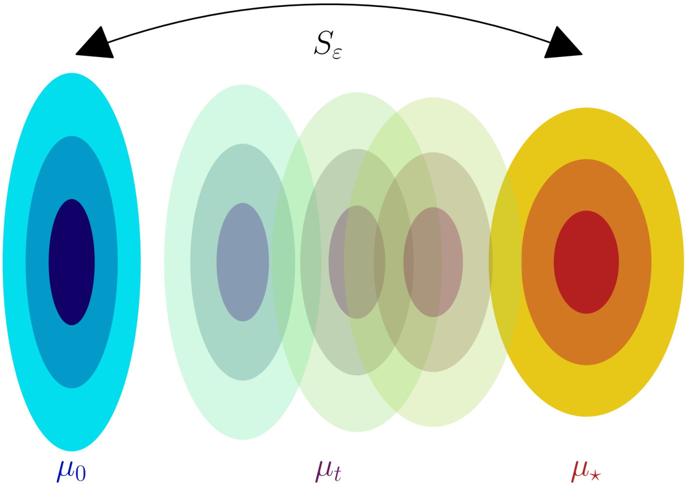
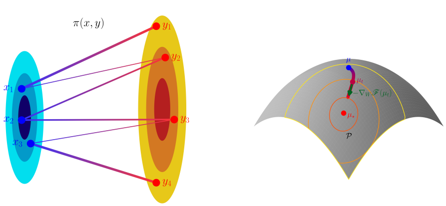


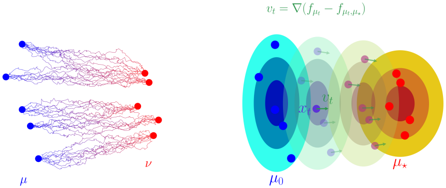
$(\mu_t)_t$ described by a PDE.

Goals: Well-posedness of that PDE, convergence criterion.

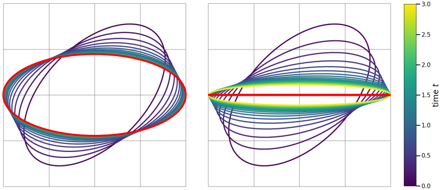
Plan



1. Optimal transport and gradient flows

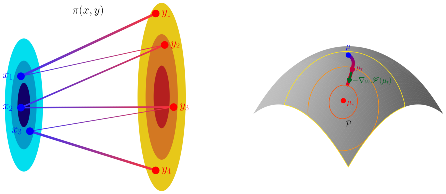


2. The Sinkhorn divergence and its flow



3. Main results

Plan



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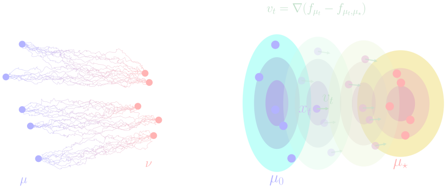

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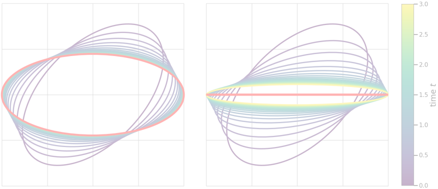
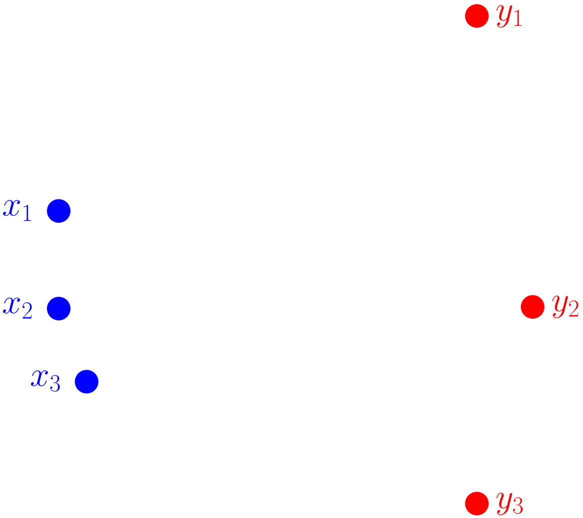


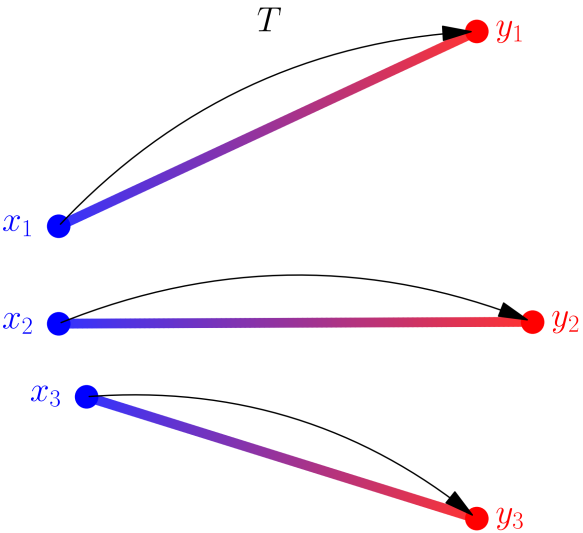
3. Main results

The Monge assignment problem



[Villani, 2009]

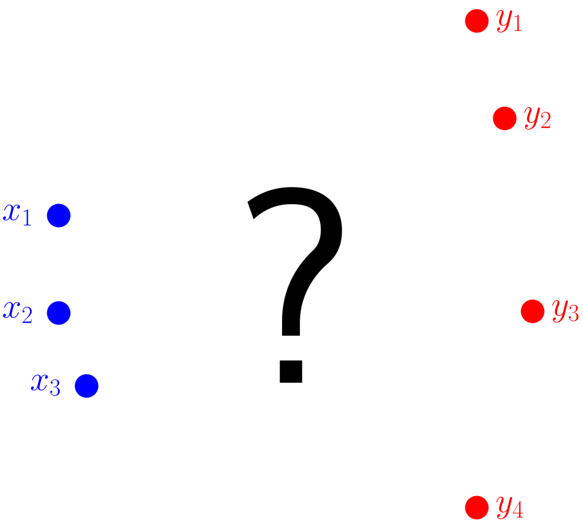
The Monge assignment problem



$$\min_{T_\# \mu = \nu} \int \|x - T(x)\|^2 \mathrm{d}\mu(x)$$

[Villani, 2009]

The Monge assignment problem



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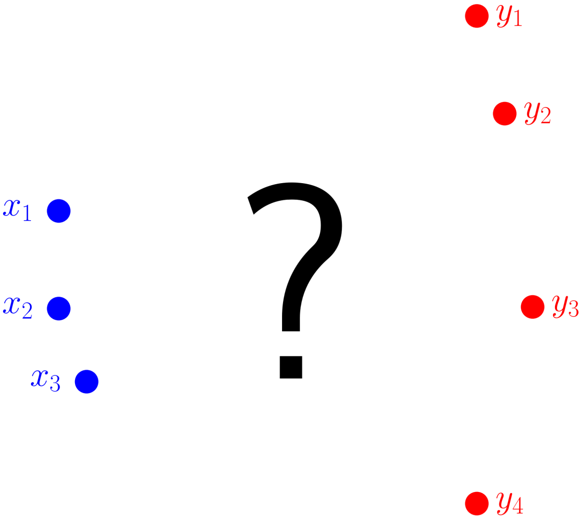
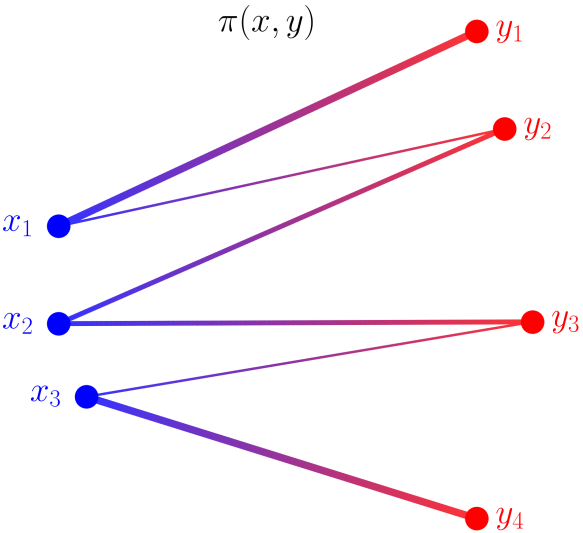


$$\min_{\boxed{T_{\#}\mu = \nu}} \int \|x - T(x)\|^2 \mathrm{d}\mu(x)$$

May be empty and is non-convex in general !

[Villani, 2009]

Kantorovitch relaxation



$$\min_{T_\# \mu = \nu} \int \|x - T(x)\|^2 \mathrm{d}\mu(x)$$

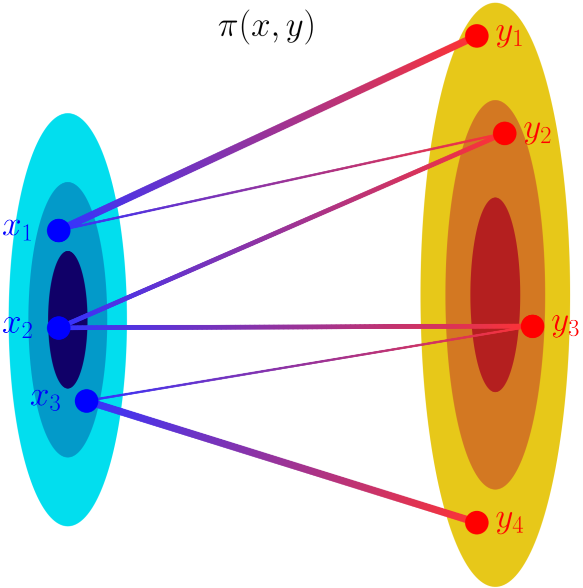
$$\Downarrow$$

$$W_2(\mu, \nu)^2 := \min_{\pi \in \Pi(\mu, \nu)} \int \|x - y\|^2 \mathrm{d}\pi(x, y)$$

↳ Set of couplings

[Villani, 2009]

Kantorovitch relaxation



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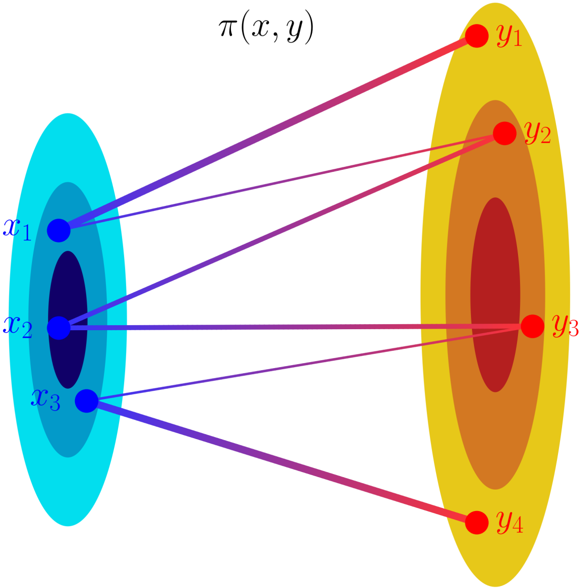

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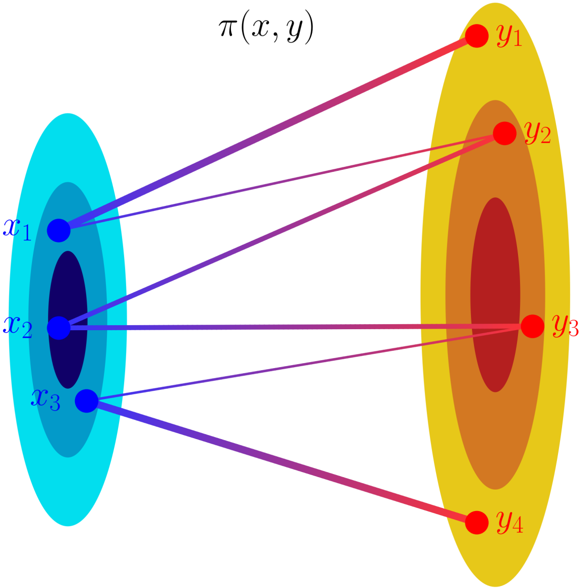

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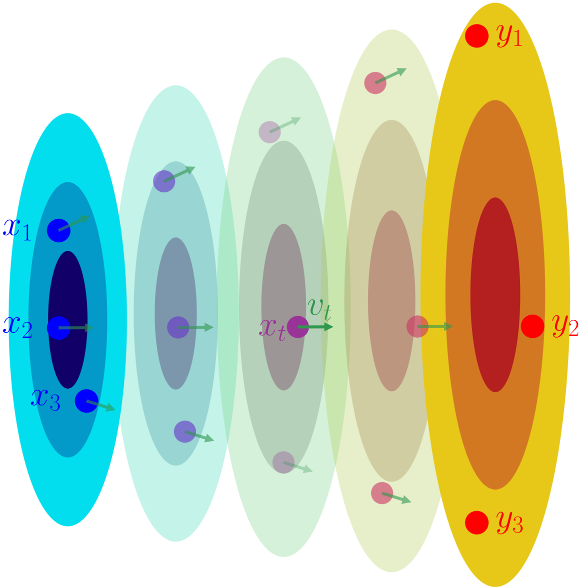
Theorem (Brenier). *When μ has a density, the Monge and Kantorovitch problems are equivalent: there is a unique Monge map $T = \mathrm{Id} - \frac{1}{2}\nabla\varphi$ and the optimal Kantorovitch plan is $(\mathrm{Id}, T)_\# \mu$.*

[Brenier, 1991]

Benamou-Brenier dynamical formulation

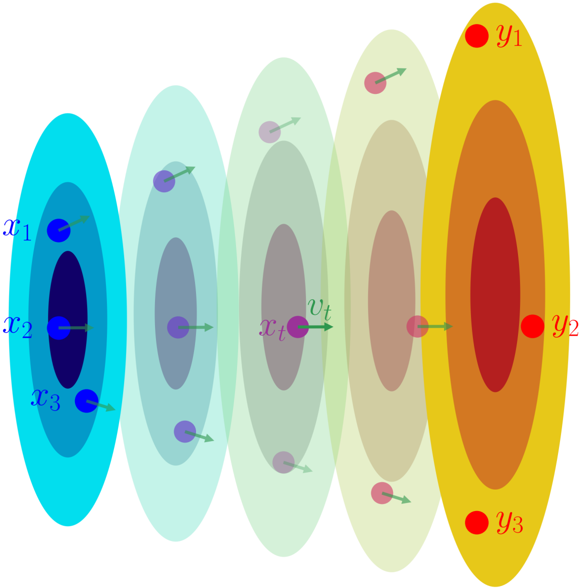


$$W_2(\mu,\nu)^2 = \min \int_0^1 \int_{\mathbb{R}^d} \|v_t(x)\|^2 \mathrm{d}\mu_t(x) \mathrm{d}t$$

$$\text{over } \dot{\mu}_t + \mathrm{div}(\mu_t v_t) = 0,\ \mu_0 = \mu,\ \mu_1 = \nu.$$

[Benamou & Brenier, 2000]

Benamou-Brenier dynamical formulation



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Intuitively: particles follow $\dot{x}_t = v_t(x_t)$.
All A.C. curves follow such an evolution.

[Benamou & Brenier, 2000]

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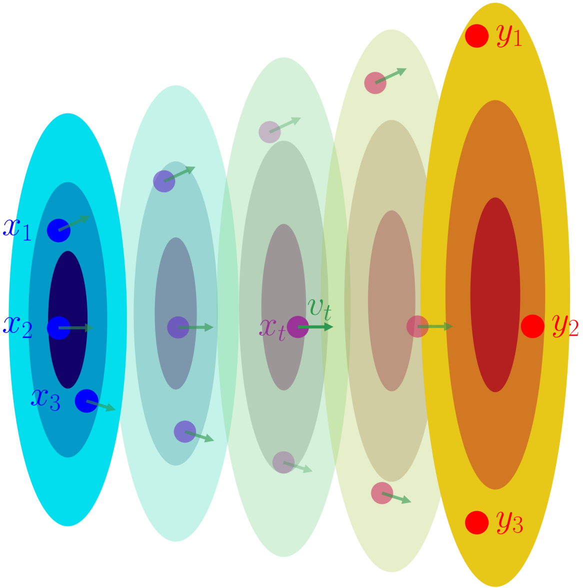

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Intuitively: particles follow $\dot{x}_t = v_t(x_t)$.
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If T exists, $v_t = T - \mathrm{Id}$ for all t.

[Benamou & Brenier, 2000]

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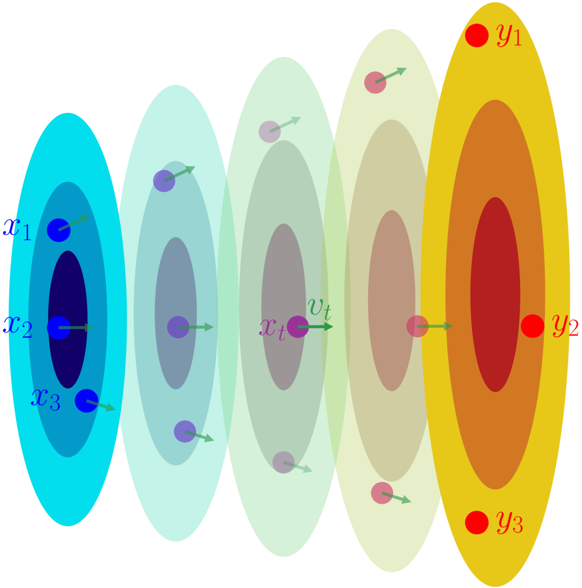

$$W_2(\mu,\nu)^2 = \min \int_0^1 \|v_t\|^2_{L^2_{\mu_t}} \,\mathrm{d}t$$

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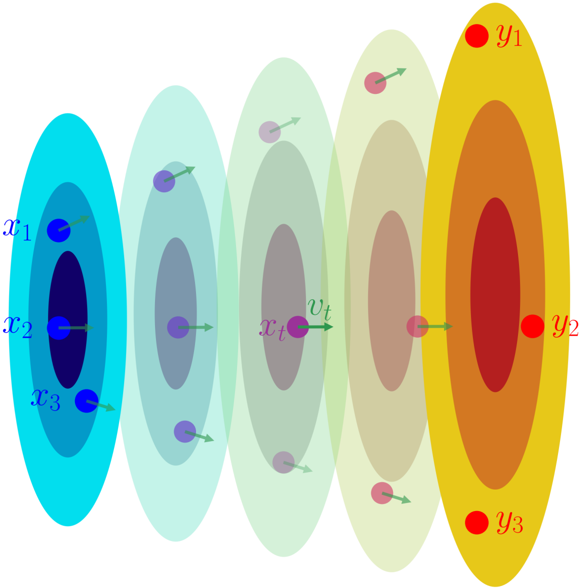

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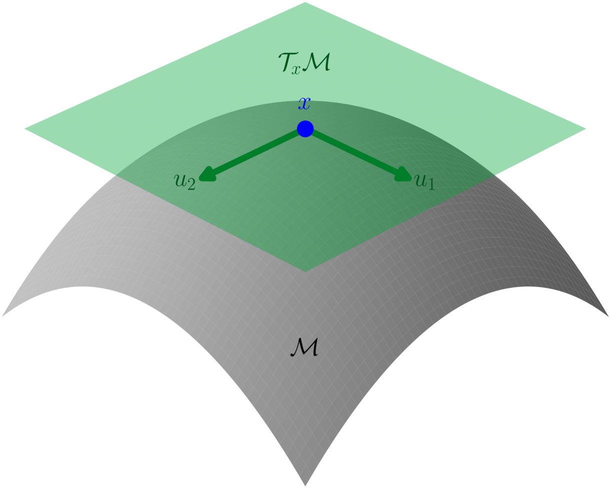
Looks like a formula from Riemannian geometry...

[Benamou & Brenier, 2000]

Riemannian geometry in the Wasserstein space



[Ambrosio, Gigli & Savaré, 2008]

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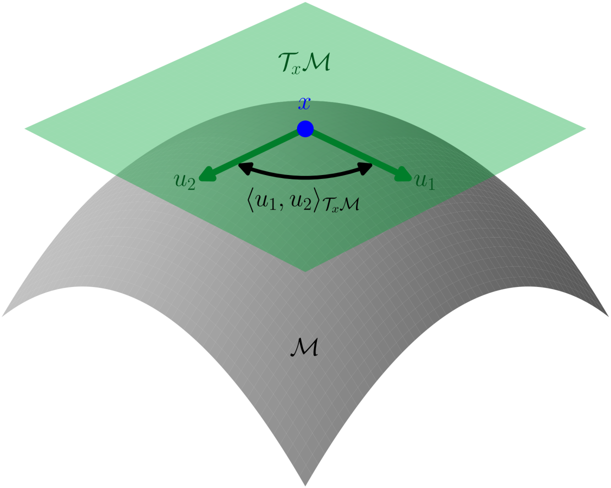

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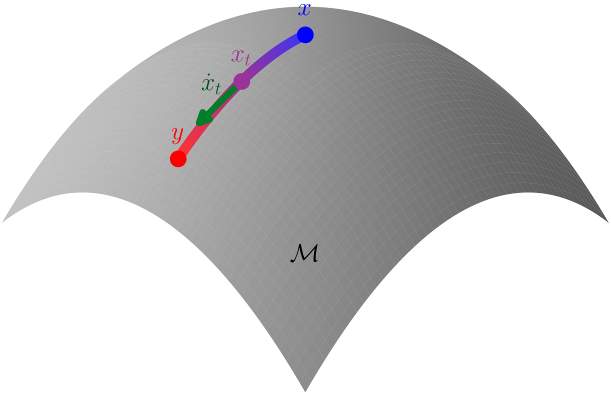

$$\mathsf{d}(x, y)^2 = \min \int_0^1 \|\dot{x}_t\|^2_{\mathcal{T}_{x_t}\mathcal{M}} \mathrm{d}t$$

over $(x_t)_t \subset \mathcal{M}$ such that $x_0 = x$, $x_1 = y$

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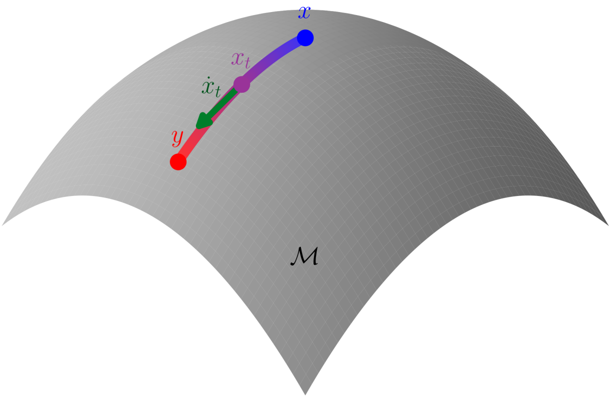

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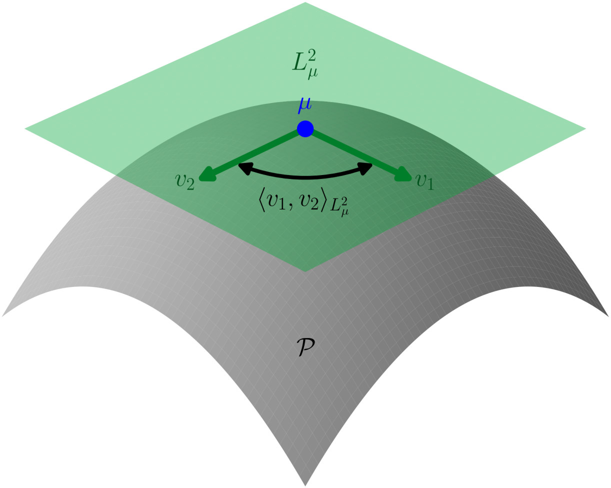

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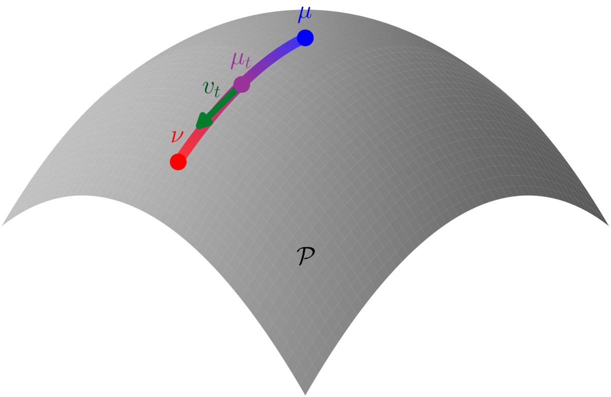

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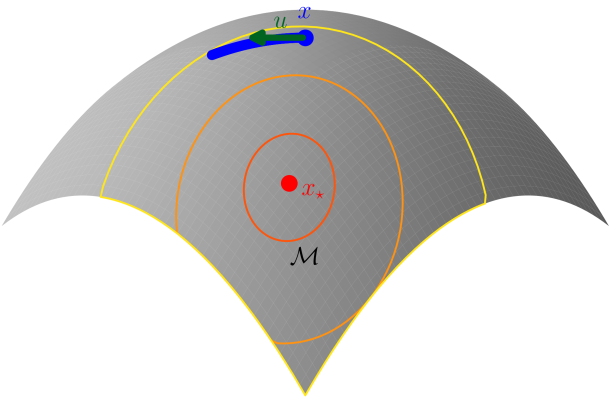
Wasserstein Gradient flows

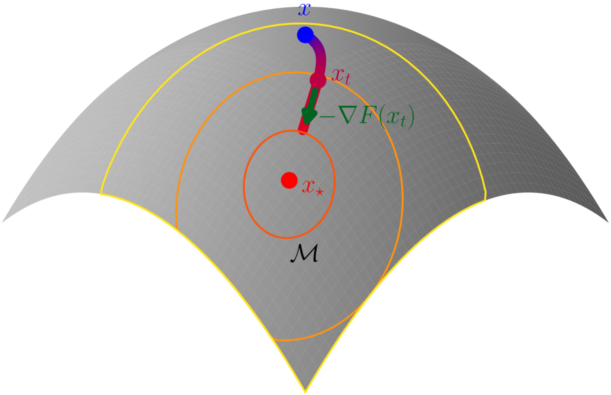


$$\text{For } F : \mathcal{M} \to \mathbb{R},\ \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} F(x_s^u) = \langle \nabla F(x), u \rangle_{\mathcal{T}_x\mathcal{M}}$$

$$\text{with } x_0^u = x \text{ and } \dot{x}_0^u = u$$

[Otto, 2001], [Ambrosio, Gigli & Savaré, 2008]

Wasserstein Gradient flows



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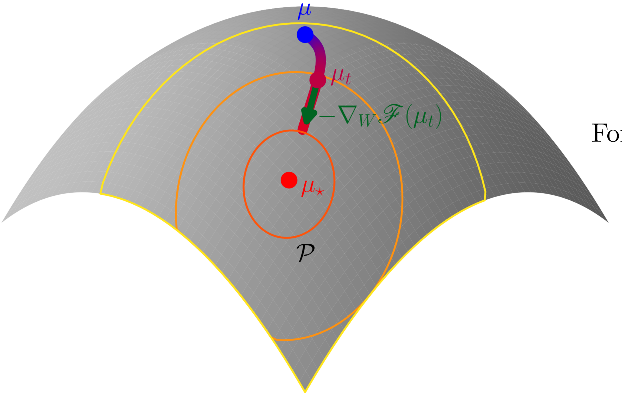
with $x_0^u = x$ and $\dot{x}_0^u = u$

Gradient flow: $\dot{x}_t = -\nabla F(x_t)$

[Otto, 2001], [Ambrosio, Gigli & Savaré, 2008]

Wasserstein Gradient flows



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For $\mathscr{F} : \mathcal{P} \to \mathbb{R}$, $\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathscr{F}(\mu_s^v) = \langle \nabla_W \mathscr{F}(\mu), v \rangle_{L^2_\mu}$

with $\mu_0^v = \mu$ and $\dot{\mu}_0^v + \mathrm{div}(\mu v) = 0$

Gradient flow: $\begin{aligned} &\dot{\mu}_t + \mathrm{div}(\mu_t v_t) = 0, \\ &v_t = -\nabla_W \mathscr{F}(\mu_t) \end{aligned}$

[Otto, 2001], [Ambrosio, Gigli & Savaré, 2008]

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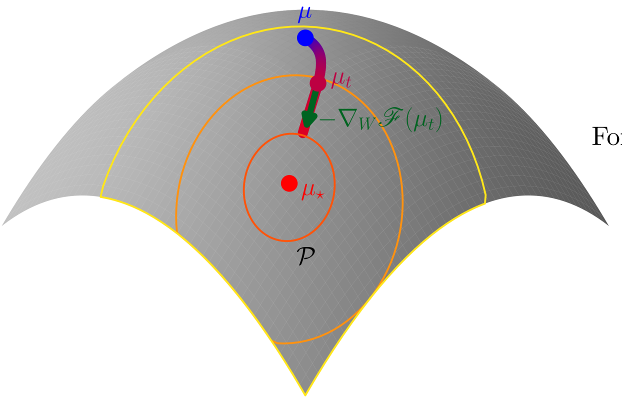

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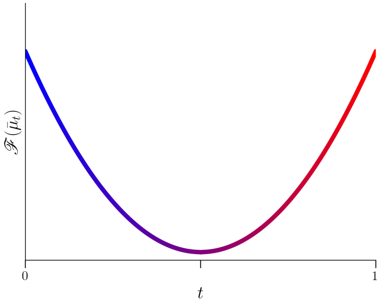
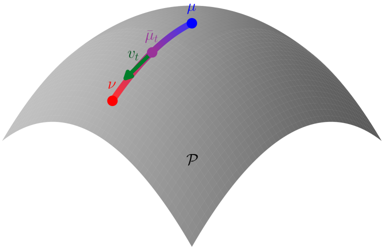
Gradient flow: $\begin{aligned} \dot{\mu}_t + \mathrm{div}(\mu_t v_t) &= 0, \\ v_t &= -\nabla_W \mathscr{F}(\mu_t) \end{aligned}$

Usually, $\nabla_W \mathscr{F} = \nabla \frac{\delta \mathscr{F}}{\delta \mu}$

[Otto, 2001], [Ambrosio, Gigli & Savaré, 2008]

λ-convexity and EVI





$$\mathscr{F}(\bar{\mu}_t) \leq (1-t)\mathscr{F}(\mu) + t\mathscr{F}(\nu)$$

[Ambrosio, Gigli & Savaré, 2008]

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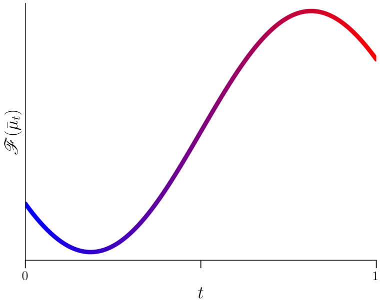

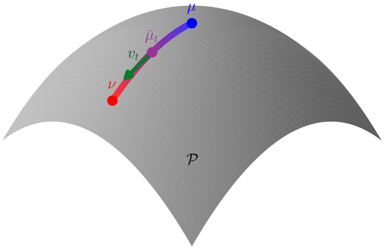

$$\mathscr{F}(\bar{\mu}_t) \leq (1-t)\mathscr{F}(\mu) + t\mathscr{F}(\nu) - \frac{\lambda}{2}t(1-t)W_2^2(\mu,\nu)$$

[Ambrosio, Gigli & Savaré, 2008]

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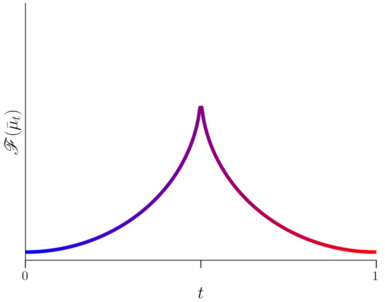

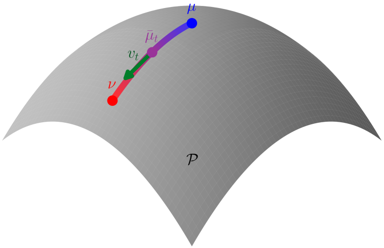

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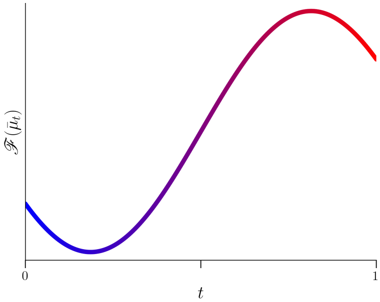

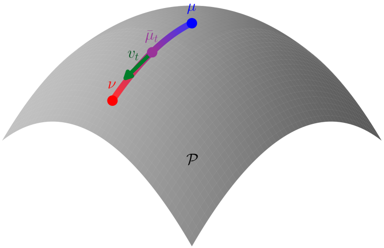

$$\mathscr{F}(\bar{\mu}_t) \leq (1-t)\mathscr{F}(\mu) + t\mathscr{F}(\nu) - \frac{\lambda}{2}t(1-t)W_2^2(\mu, \nu)$$

For a gradient flow $(\mu_t)_t$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t, \nu) \leq \mathscr{F}(\nu) - \mathscr{F}(\mu_t) - \frac{\lambda}{2}W_2^2(\mu_t, \nu)$$

[Ambrosio, Gigli & Savaré, 2008]

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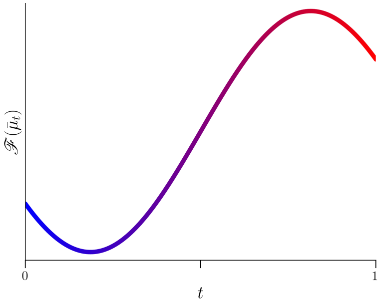
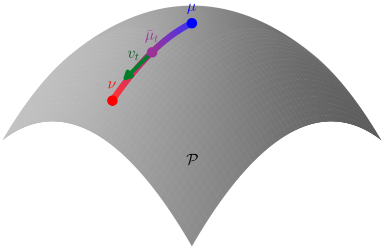




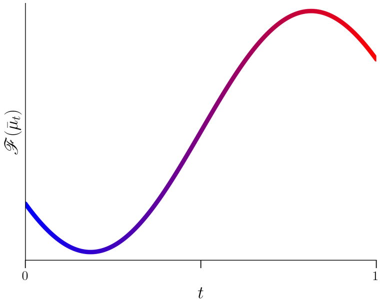
$$\mathscr{F}(\bar{\mu}_t) \leq (1-t)\mathscr{F}(\mu) + t\mathscr{F}(\nu) - \frac{\lambda}{2}t(1-t)W_2^2(\mu,\nu)$$

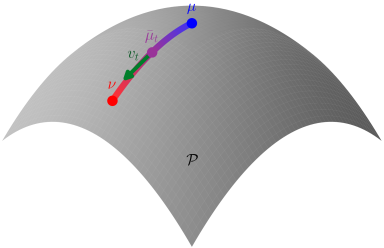
For a gradient flow $(\mu_t)_t$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t,\nu) \leq \mathscr{F}(\nu) - \mathscr{F}(\mu_t) - \frac{\lambda}{2}W_2^2(\mu_t,\nu)$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t^1,\mu_t^2) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t}W_2^2(\mu_s^1,\mu_t^2) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t}W_2^2(\mu_t^1,\mu_s^2) \\ &\leq -\lambda W_2^2(\mu_t^1,\mu_t^2)\end{aligned}$$

[Ambrosio, Gigli & Savaré, 2008]





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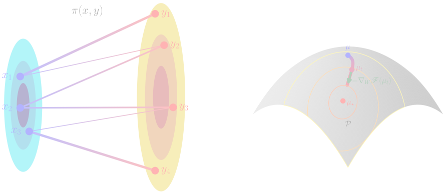
$$\implies W_2^2(\mu_t^1,\mu_t^2) \leq e^{-2\lambda t}W_2^2(\mu_0^1,\mu_0^2)$$

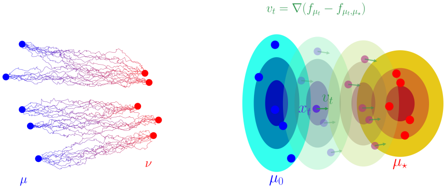
yielding uniqueness. Existence is also given by AGS.

[Ambrosio, Gigli & Savaré, 2008]

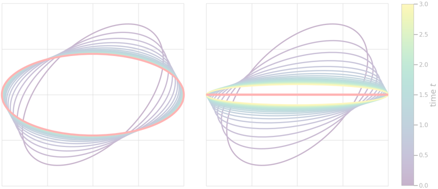
Plan



1. Optimal transport and gradient flows

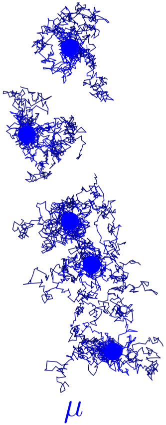


2. The Sinkhorn divergence and its flow



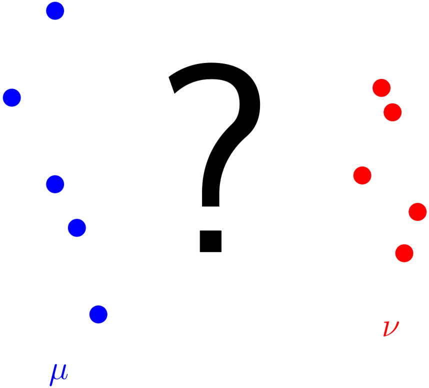
3. Main results

Entropic optimal transport



[Schrödinger, 1931], [Léonard, 2014]

Entropic optimal transport



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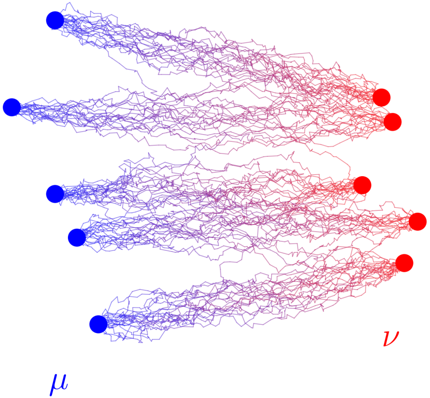

Dynamic Schrödinger problem:

$$\min_{P_0=\mu, P_1=\nu} \mathrm{KL}(P|R)$$

where P is a distribution of paths, R is the Brownian motion of diffusivity $\varepsilon > 0$.

[Schrödinger, 1931], [Léonard, 2014]

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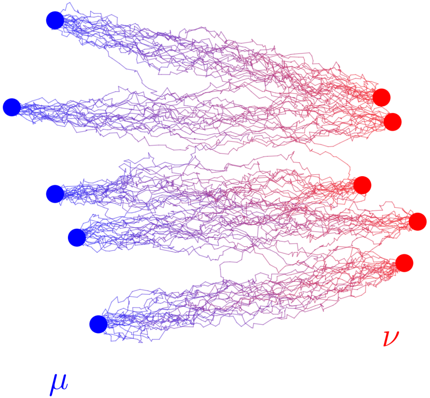

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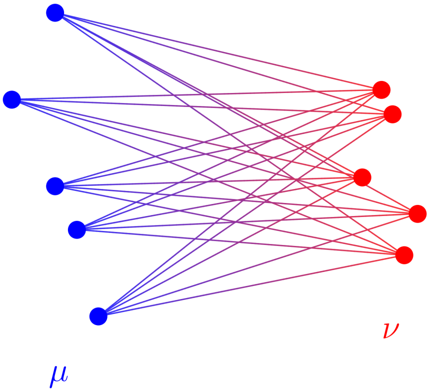
Entropic optimal transport problem:

$$\min_{\pi \in \Pi(\mu,\nu)} \int \|x - y\|^2 \mathrm{d}\pi(x, y) + \varepsilon \mathrm{KL}(\pi|\mu \otimes \nu)$$

[Schrödinger, 1931], [Léonard, 2014]

Entropic optimal transport



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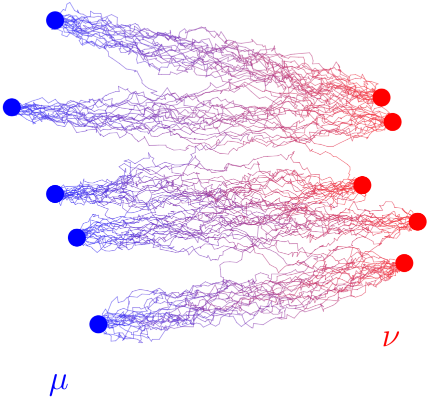

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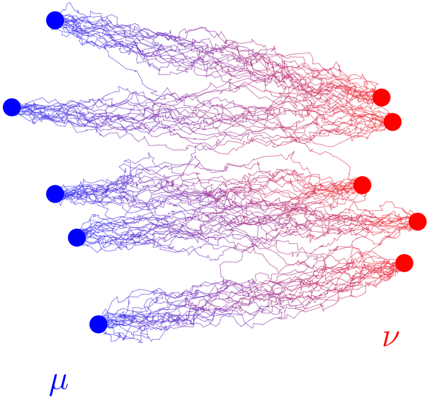
$\pi(x, y)$ is the mass that goes from x to y in the Schrödinger bridge.

[Schrödinger, 1931], [Léonard, 2014]

Dual problem and barycentric map



$$\mathrm{OT}_\varepsilon(\mu,\nu) := \min_{\pi\in\Pi(\mu,\nu)} \int \|x-y\|^2 \mathrm{d}\pi(x,y) + \varepsilon \mathrm{KL}(\pi|\mu\otimes\nu)$$

[Peyré & Cuturi, 2018]

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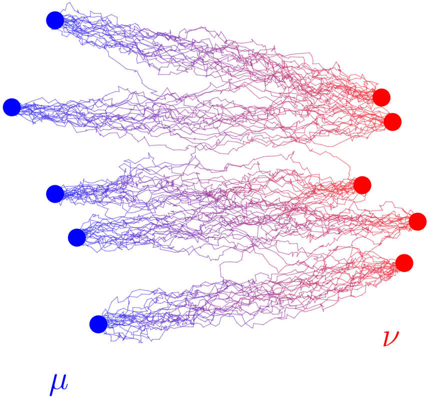

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[Peyré & Cuturi, 2018]

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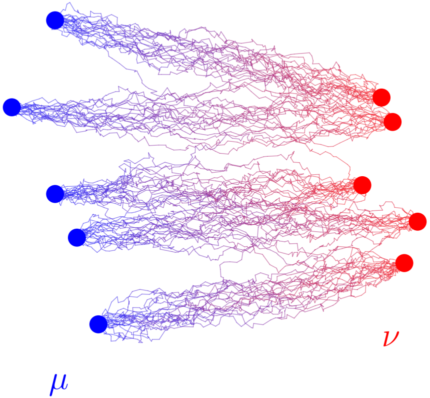

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Recall: $W_2^2(\mu,\nu) = \max_{\varphi\oplus\psi\leq c} \int \varphi\mathrm{d}\mu + \int \psi\mathrm{d}\nu$

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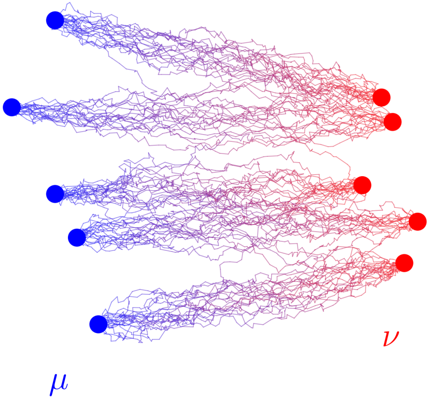

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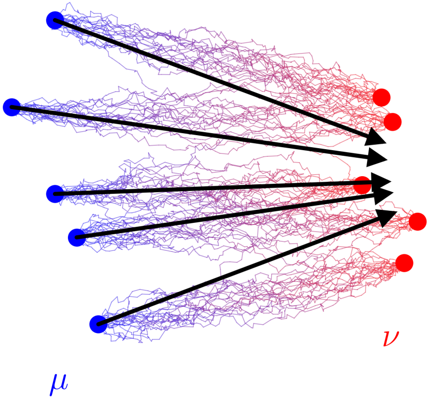
[Peyré & Cuturi, 2018]

Dual problem and barycentric map

$T^{\varepsilon}_{\mu,\nu} - \mathrm{Id}$



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Writing $f_{\mu,\nu}, g_{\mu,\nu}$ the Schrödinger potentials,

$$T^{\varepsilon}_{\mu,\nu} := \mathrm{Id} - \tfrac{1}{2}\nabla f_{\mu,\nu}$$

[Pooladian & Niles-Weed, 2021]

Dual problem and barycentric map

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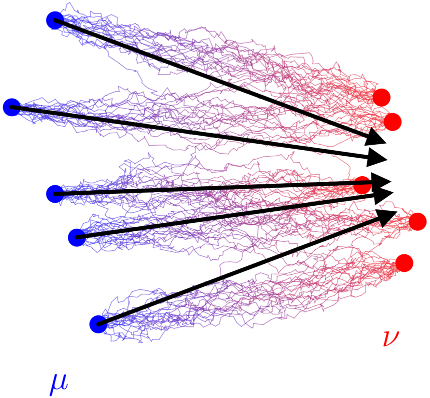

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$$x \mapsto \int y \underbrace{p_{\mu,\nu}(x,y)}_{\propto \pi(x,y)} \mathrm{d}\nu(y)$$

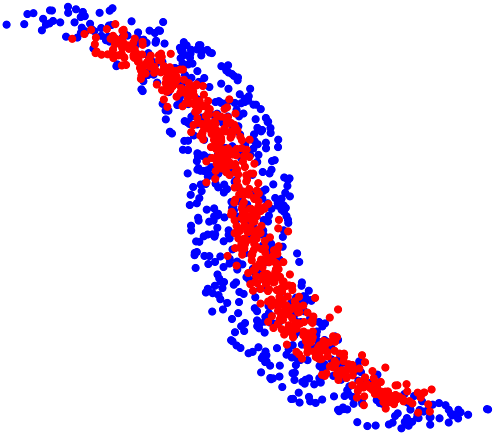
[Pooladian & Niles-Weed, 2021]

The Sinkhorn divergence

Advantages of OT_ε:

- Computed efficiently (Sinkhorn's algorithm)
- Retains the geometric flavour of W_2

The Sinkhorn divergence



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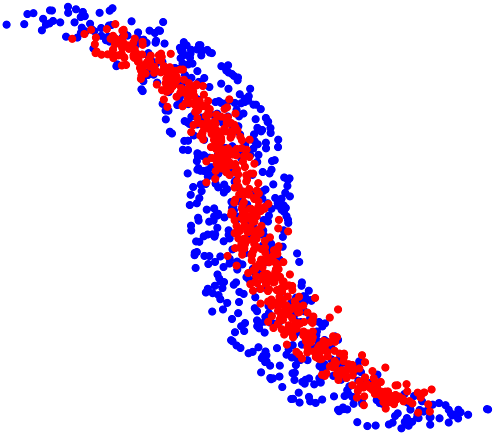

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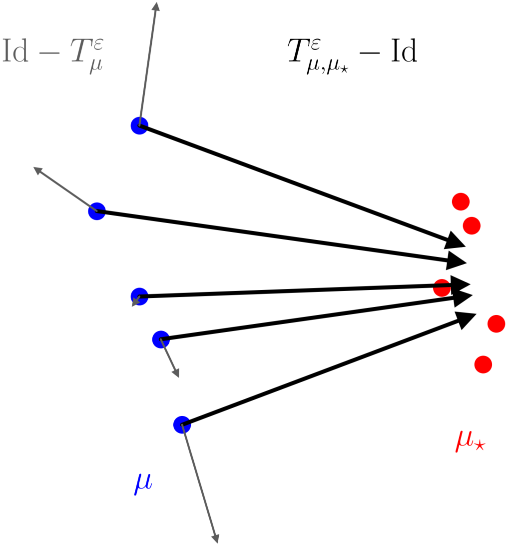
$\downarrow$

Solution: simply subtract bias !

$$S_\varepsilon(\mu,\nu) := \mathrm{OT}_\varepsilon(\mu,\nu) - \tfrac{1}{2}\mathrm{OT}_\varepsilon(\mu,\mu) - \tfrac{1}{2}\mathrm{OT}_\varepsilon(\nu,\nu)$$

[Genevay, Peyré, Cuturi, 2018] [Feydy, Séjourné, Vialard, Amari, Trouvé, Peyré, 2019]

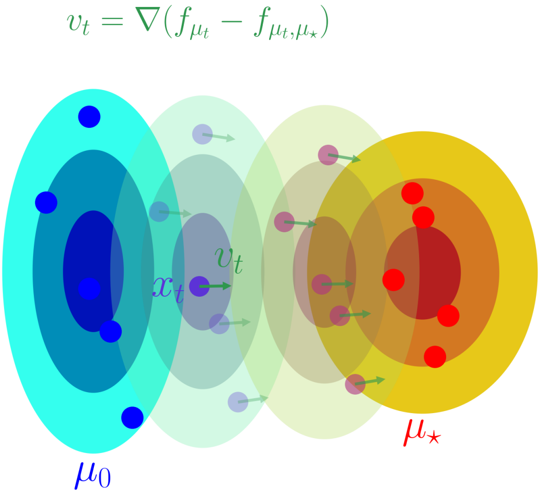
Wasserstein gradient flow of S_ε



We *should* have

$$
\begin{aligned}
-\nabla_W^1 S_\varepsilon(\mu, \mu_\star) &= \nabla(f_\mu - f_{\mu,\mu_\star}) \\
&= 2(T_{\mu,\mu_\star}^\varepsilon - \mathrm{Id} + \mathrm{Id} - T_\mu^\varepsilon)
\end{aligned}
$$

Wasserstein gradient flow of S_ε



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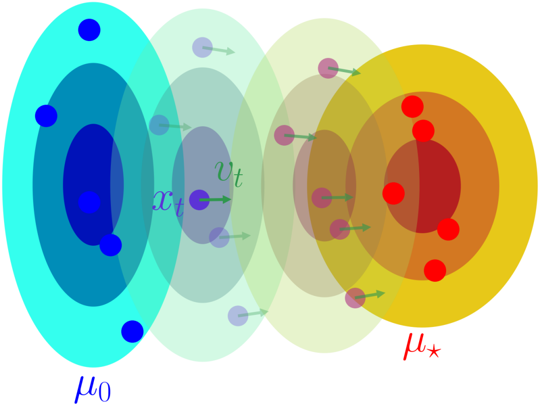
The gradient flow *should* be described by

$$\dot{\mu}_t + \operatorname{div}(\mu_t \nabla(f_{\mu_t} - f_{\mu_t,\mu_\star})) = 0$$

Wasserstein gradient flow of S_ε

$$v_t = \nabla(f_{\mu_t} - f_{\mu_t,\mu_\star})$$



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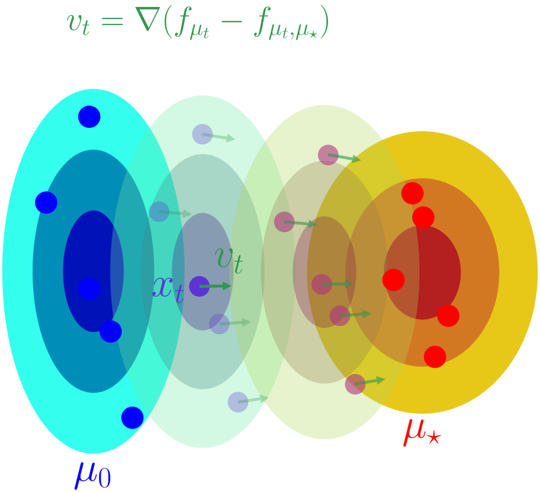
Working assumptions: $\mu_0, \mu_\star$ Gaussian

[Janati, Muzellec, Peyré, Cuturi, 2020]

Wasserstein gradient flow of S_ε



We *should* have

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Working assumptions: $\mu_0, \mu_\star$ Gaussian

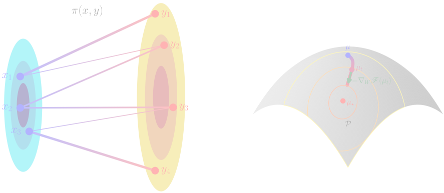
For $\mu = \mathcal{N}(m,\Sigma), \nu = \mathcal{N}(n,\Gamma)$, we have

$$S_\varepsilon(\mu,\nu) = \|m-n\|^2 + B_\varepsilon(\Sigma,\Gamma)$$

$\longrightarrow$ we can consider centered measures.

[Janati, Muzellec, Peyré, Cuturi, 2020]

Plan



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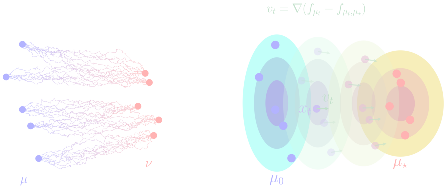

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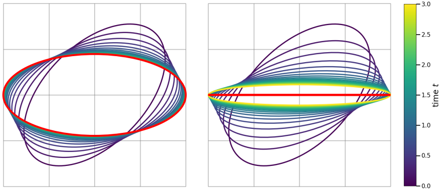

3. Main results

Well-posedness

Theorem (H & Lacombe, 2026). *Take μ_0, $\mu_\star$ Gaussian (can be singular).*
There exists a solution to

$$\dot{\mu}_t + \mathrm{div}(\mu_t \nabla (f_{\mu_t} - f_{\mu_t,\mu_\star})) = 0$$

which stays Gaussian.
It is unique among Gaussians and in a larger class $\mathcal{R} = \{\exp(-V), \alpha_V I \preceq \nabla^2 V \preceq \beta_V I\}$.
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- Constrain S_ε to Gaussians by adding $\iota_{\mathcal{G}}$.

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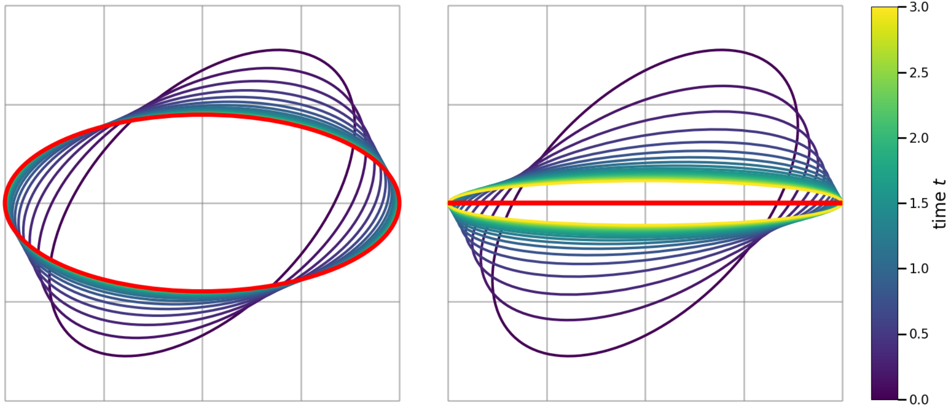
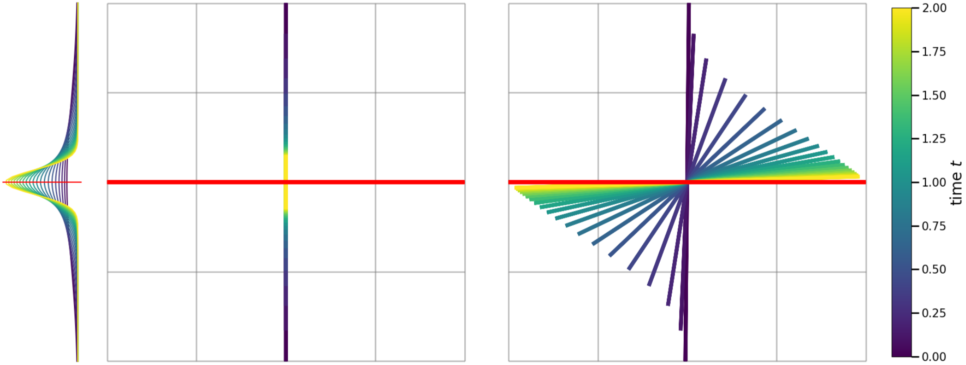
- Show $\nabla_W^\mu S_\varepsilon(\mu, \mu_\star) = \nabla(f_{\mu,\mu_\star} - f_\mu)$.
- For uniqueness in $\mathcal{R}$, get EVI with Entropic Cafarelli [Chewi & Pooladian, 2023]

Convergence



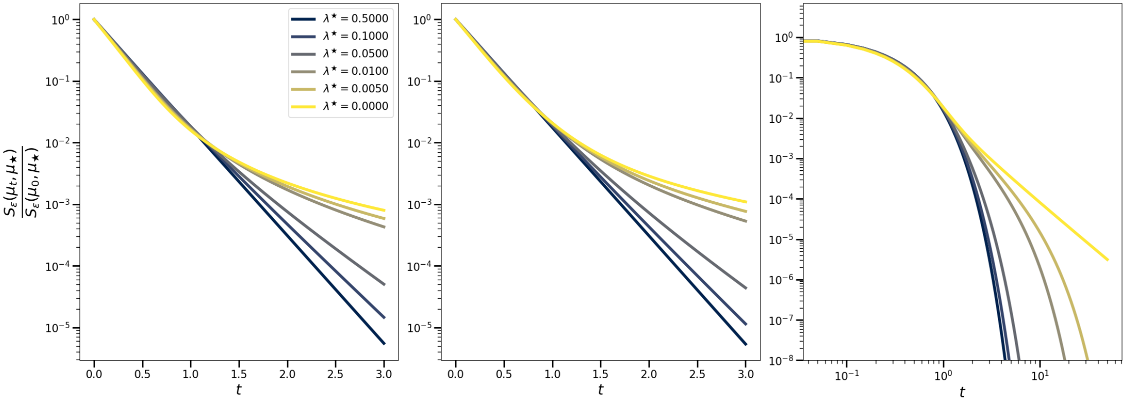
Theorem (H & Lacombe, 2026). *If μ_0 is non-singular, $\mu_t \xrightarrow[t\to\infty]{} \mu_\star$.*

Convergence



Theorem (H & Lacombe, 2026). *μ_0 is singular $\iff$ $\forall t, \mu_t$ also singular.*
$\Sigma_t \xrightarrow[t\to\infty]{} \Sigma_\infty = P\mathrm{diag}((\lambda_i)_i)P^T$ *where* $\Sigma_\star = P\mathrm{diag}((\lambda_i^\star)_i)P^T$ *and* $\lambda_i \in \{0, \lambda_i^\star\}$

Convergence



Theorem (H & Lacombe, 2026). *If Σ_0 and $\Sigma_\star$ commute, convergence holds iff $\mathrm{supp}(\mu_\star) \subset \mathrm{supp}(\mu_0)$, in $O(e^{-Ct})$ if equality and $O(\frac{1}{t})$ otherwise.*

Conclusion

What we saw:

- The Wasserstein space and its geometry, gradient flows
- Entropic optimal transport, the Sinkhorn divergence and its flow
- First convergence properties for Gaussians

Next steps:

- Particle case study (ongoing)
- More general results: convergence criterion related to existence and uniqueness of Monge maps ?

Thank you for your attention !

Appendix

Convergence rate as function of ε