Project Report: Mean Curvature Motion of Point Cloud Varifolds

Mathis Hardion
mathis.hardion@telecom-paris.fr
MVA, Télécom Paris
Palaiseau, France

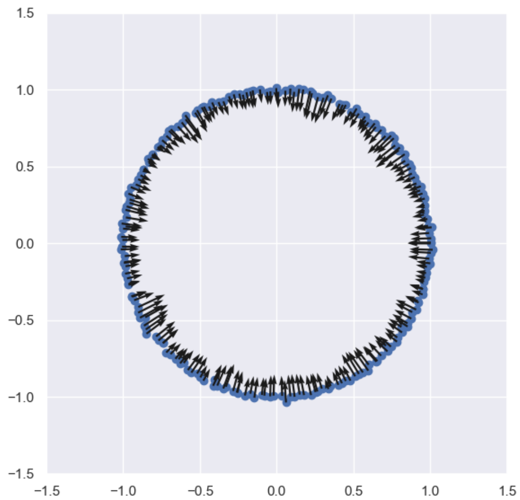

Figure 1: Mean curvature motion of a noisy circle

ABSTRACT

The paper studied in this report builds upon earlier work to develop a time discretization of mean curvature flow in the case of point cloud varifolds. A generalized version of an approximation of mean curvature is proposed and convergence bounds are derived. A semi-implicit numerical scheme is built and desirable properties such as planar barriers and sphere comparisons hold in certain cases. This report highlights theoretical and practical limitations to these extensions, and suggests directions of further improvement.

KEYWORDS

Point Cloud Varifolds, Mean Curvature Motion, Time Discretization

1 INTRODUCTION

This report discusses the context, implications and limits of [7] and aims to show the underlying concepts and results under a slightly different light.[1] The report is organized as follows. Section 2 reintroduces the background and notions upon which the authors of [7] build their contributions. Section 3 critically addresses the main contributions of the paper, their derivations and limitations. Section 4 proposes some directions in which the developed framework could be refined and expanded upon. Conclusively, section 5 makes some final remarks about the paper.

[1]The author of this report has mostly studied fields related to probability theory, explaining the ensuing point of view.

2 CONTEXT

2.1 Preceding work, motivations

This work builds upon the field of Geometric Measure Theory (GMT), originally developed as an approach to solve Plateau's problem of minimal area surfaces given boundary conditions, formulated by Lagrange in 1760. The notion of varifold is attributed to Almgren [2], who introduced a measure-theoretic point of view on the generalized surfaces earlier defined by Young [12], as a model for soap films in the study of this problem. As we shall see, this definition generalizes that of a surface and is advantageous when considering multiplicity. It also allows for generalization of the mean curvature vector, which is relevant to Plateau's problem as it gives the direction in which the area decreases the most. Therefore, analogously to gradient descent, moving each point of a surface according to this direction will converge towards the minimal surface. This evolution is known as *mean curvature motion* or *curvature flow*. Despite the historical origin of these concepts, they are very general and apply to the study of all shapes/surfaces, which is of importance in various fields such as physics, biology (e.g. molecules), medicine (e.g. anatomy), architecture and more. A notable interest to the study of mean curvature motion are the smoothing of surfaces/meshes [9], considering they are often digitalized from laser scans which output potentially noisy point clouds. A difficulty in that case is the reconstruction of the mean curvature, and the main author of the paper studied in this report has proposed the varifold framework as an answer in preceding works [4–6]. As of the making of this report, it appears such a framework is still relatively scarcely utilized by other authors in computational geometry or shape analysis, with a few notable contributions in [8] which employs varifolds in the context of computational anatomy furthered in [11] for shape comparison and registration. The explanation of this unpopularity given by [6] is firstly that varifold theory comprises very technical results creating a certain barrier to entry, and secondly that its development was not driven by discrete considerations of geometric objects. The main author therefore aims to show relevance of GMT in the current landscape of research in computational geometry as a general tool for both continuous and discrete objects.

2.2 Varifolds

The central concept of this work is that of a d-varifold, sometimes also referred to as generalized surface, formalized as follows.

Definition 2.1. A varifold V is a Radon measure on $\mathbb{R}^n \times G_{d,n}$, where $G_{d,n}$ is the set of d-dimensional subspaces of $\mathbb{R}^n$.

Under the assumption that V has compact support, which is made later by the authors to prove some results we shall see in section 3, a varifold can then be seen as a probability measure (up to a normalizing constant). Equivalently, V can then be seen as the law of a random variable (X, P) valued in $\mathbb{R}^n \times G_{d,n}$, and the meaning of "generalized surface" can then be understood as a distribution of d-dimensional "patches" defined by their location and direction. The mass of V defined as

$$\|V\|(\cdot) \coloneqq V(\cdot \times G_{d,n})$$

can then be thought of as the marginal distribution of X. In this report, we also denote the other marginal as

$$\vec{V}(\cdot) \coloneqq V(\mathbb{R}^n \times \cdot).$$

One can then embed the d-Manifolds in the space of varifolds through definition 2.2.

Definition 2.2. A smooth varifold associated to the d-submanifold $M \subset \mathbb{R}^n$ is defined by

$$V(\cdot) = \mathcal{H}^d(\{x \in M, (x, T_x M) \in \cdot\}).$$

Such a varifold is also denoted by the authors as $\mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ due to how it acts on measurable functions, however one may also write it as a pushforward measure

$$V = \pi_M^{-1} \sharp \mathcal{H}^d_{|M} \tag{1}$$

where π_M^{-1} is the "inverse tangent bundle" of M, i.e. $\pi_M^{-1} : x \mapsto (x, T_x M)$. Then, considering that the Hausdorff measure is essentially a generalization of the Lebesgue measure, one can see the smooth varifold V as a uniform distribution over the surface corresponding to manifold M. Said otherwise, we also have that

$$\vec{V} = T.M \sharp \|V\|,$$

with $T.M : x \mapsto T_x M$. The discrete counterpart to the above is then a discrete distribution, formalized as the following.

Definition 2.3. A point cloud d-varifold writes

$$V = \sum_{i=1}^{N} m_i \delta_{x_i, P_i}$$

for some $N > 0$ and with for $1 \leq i \leq N$, $m_i > 0$, $x_i \in \mathbb{R}^n$, $P_i \in G_{d,n}$.

Following [1], and writing for a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ of jacobian ∂F, and $P \in G_{d,n}$ with orthogonal projection matrixΠ_P, $\mathrm{div}_P(F)(\cdot) \coloneqq \mathrm{Tr}(\Pi_P \partial F(\cdot)) \eqqcolon \langle \Pi_P, \partial F(\cdot) \rangle$, one can define the first variation as follows.

Definition 2.4. The first variation of V is the (Schwartz) distribution

$$\delta V : \left| \begin{array}{ccl} C^1_c(\mathbb{R}^n, \mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ F & \longmapsto & \int_{\mathbb{R}^n \times G_{d,n}} \langle \Pi_P, \partial F(x) \rangle \, dV(x, P). \end{array} \right.$$

The mean curvature at a point of a manifold M, intuitively the average of the principal curvatures times the normal vector at that point, is linked with the first variation when $V = \pi_M^{-1} \sharp \mathcal{H}^d_{|M}$ by the divergence theorem. To extend this idea, the generalized mean curvature is then defined under regularity assumptions as the vector-valued function H verifying

$$\forall F \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \delta V(F) = -\int_{\mathbb{R}^n} H(x) \cdot F(x) d\|V\|(x) + \mathfrak{d}^{\perp}_{\|V\|}(F)$$

with $\mathfrak{d}^{\perp}_{\|V\|}$ is a distribution "orthogonal" to $\|V\|$ in the sense that there exist a partition of $\mathbb{R}^n$ in two borel sets A, B such that the restriction of $\mathfrak{d}^{\perp}_{\|V\|}$ to test functions of support subset of A is 0 and $\|V\|(B) = 0$. Under our probabilistic notations and for such test functions, one can understand the first variation as

$$\mathbb{E}\left[\mathrm{div}_P F(X)\right] = -\mathbb{E}\left[H(X) \cdot F(X)\right].$$

2.3 Approximate mean curvature

The paper extends the notion of *approximate mean curvature* defined in [6], which was initially developed as a way to define a notion of curvature without any regularity assumptions. This is made through convolution with smooth kernels

$$\rho_\varepsilon(\cdot) = \varepsilon^{-n} \rho\left(\frac{\|\cdot\|}{\varepsilon}\right),\ \xi_\varepsilon = \varepsilon^{-n} \xi\left(\frac{\|\cdot\|}{\varepsilon}\right)$$

where ρ, ξ are smooth compacted in $[-1, 1]$ and $\varepsilon > 0$, meaning the kernels essentially converge to a dirac distribution when $\varepsilon \to 0$. The *regularized first variation* is then $\delta V * \rho_\varepsilon \coloneqq \delta V(\cdot * \rho_\varepsilon)$, and the regularized mass is $\|V\| * \xi_\varepsilon \coloneqq \int \xi_\varepsilon(y - \cdot)\, d\|V\|(y)$. With specific assumptions on ξ and ρ, one can obtain a density of the regularized first variation with respect to the regularized mass, giving rise to the following definition.

Definition 2.5. For a d-varifold V, denoting Ψ_ε the density of $\delta V * \rho_\varepsilon$ with respect to $\|V\| * \xi_\varepsilon$, the *ε-approximate mean curvature* is defined as

$$H_\varepsilon(\cdot, V) \coloneqq -\frac{d}{n} \Psi_\varepsilon(\cdot).$$

The constant $\frac{d}{n}$ is due to the choice of ξ, ρ in light of better numerical results understood through taylor expansion in ε as detailed in [6]. Therein is proven that one recovers convergence to the mean curvature in the case $V = \pi_M^{-1} \sharp \mathcal{H}^d_{|M}$ for $\varepsilon \to 0$.

2.4 Mean curvature flow

The mean curvature flow for varifolds was firstly studied by Brakke [3], introducing a quite general framework for dynamical systems driven by surface tension. The so-called Brakke flow was originally defined to the effect of the following.

Definition 2.6. A family $(V_t)_{t\in\mathbb{R}_+}$ of d-varifolds is *moving by its mean curvature* if

$$\forall t \geq 0, \forall F \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \frac{d}{dt}\|V_t\|(F) \leq \delta V_t(F).$$

Some existence and uniqueness properties for Brakke flow were later investigated in [10]. However, such properties do not hold in general when singularities (e.g. triple points) appear. Thus, multiple schemes for mean curvature flow can be investigated.

3 MAIN CONTRIBUTIONS

3.1 Extension of approximate curvature for smooth varifold and consistency

The authors first propose mild variations of approximate curvature, seemingly not strongly motivated as they explicit that it has no real advantage and only serves to show different behaviors. More precisely, one can compute a density with respect to Lebesgue of the regularized first variation as

$$g_\varepsilon : x \mapsto \frac{1}{\varepsilon^{n+1}} \int_{\mathbb{R}^n \times G_{d,n}} \rho'\left(\frac{\|y-x\|}{\varepsilon}\right) \Pi_P \left(\frac{y-x}{\|y-x\|}\right) dV(y,P). \tag{2}$$

The authors propose replacing the projector Π_P in (2) with different possiblities of projectors Π depending on $x, y \in \mathbb{R}^n$ and $P \in G_{d,n}$, i.e. define

$$H_\epsilon^\Pi(x,V) = -\frac{d}{n\varepsilon} \frac{\displaystyle\int_{\mathbb{R}^n \times G_{d,n}} \rho'\left(\frac{\|y-x\|}{\varepsilon}\right) \frac{\Pi(y-x)}{\|y-x\|}\, dV(y,S)}{\displaystyle\int_{\mathbb{R}^n} \xi\left(\frac{\|y-x\|}{\varepsilon}\right) d\|V\|(y)}. \tag{3}$$

Then, they obtain the following proposition which we next discuss.

PROPOSITION 3.1. *For $d = n-1$, $M \subset \mathbb{R}^n$ a C^2 d-manifold of mean curvature vector $H : M \to \mathbb{R}^n$, $V = \pi_M^{-1}\sharp\mathcal{H}^d_{|M}$, and for $\Pi \in \left\{\Pi_P, -2\Pi_{P^\perp}, 2Id, \Pi_{(T_xM)^\perp} \circ \Pi_P, -2\Pi_{(T_xM)^\perp} \circ \Pi_{P^\perp}, 2\Pi_{(T_xM)^\perp}\right\}$, then for any $x \in M$,*

$$H_\varepsilon^\Pi(x,V) \xrightarrow[\varepsilon\to 0]{} H(x).$$

Additionally, if M is of class C^3, $\left|H_\varepsilon^\Pi(x,V) - H(x)\right| = O(\varepsilon)$.

It is apparent that this generalization is very partial as the result is proven only for 6 specific projectors, which are not really given meaning by the authors and the provided proof is not constructive either as it only checks through brute computation that they indeed verify the above property. Additionally, while the result is claimed to hold for any codimension, it is only proven for codimension 1 for simplification. Overall, this result and the way it is derived seem to illustrate some of the previously discussed reasons for the unpopularity of the varifold framework: heavy computations are required to make only a tiny step of generalization.

3.2 Stability of approximate curvature

Next, the authors investigate stability with respect to the varifold and location, i.e. want to show convergence of $H_\varepsilon^\Pi(z,W)$ to $H(x,V)$ when $\varepsilon \to 0$, $z \to x$, and $W \to V$ in the sense of weak-* convergence of measures. To this end, the paper works under the assumption of d-regularity as defined below.

Definition 3.2. A d-varifold V is d-regular if there exist $C_0 \geq 1$ and $r_0 > 0$ such that

$$\forall x \in \operatorname{supp}\|V\|, \forall 0 < r \leq r_0, \frac{r^d}{C_0} \leq \|V\|(\mathcal{B}(x,r)) \leq C_0 r^d.$$

Note that the above definition holds in the case of a smooth varifold $V = \pi_M^{-1}\sharp\mathcal{H}^d_{|M}$ from the properties of the Hausdorff measure. To achieve the desired result, two distances are utilized by the authors, namely the flat distance and a modified Prokhorov distance defined below.

Definition 3.3. The *flat* or *localized bounded lipshitz distance* on an open U of a locally compact separable metric space $\mathbb{X}$ between Radon measures μ and ν is

$$\Delta_U(\mu,\nu) := \sup\left\{\int_{\mathbb{X}} \varphi d(\mu-\nu) \,\middle|\, \begin{smallmatrix}\varphi \text{ 1-Lipschitz,}\\ \sup_{\mathbb{X}}|\varphi|\leq 1,\\ \operatorname{supp}\varphi\subset U\end{smallmatrix}\right\}.$$

Definition 3.4. The modified *d-Prokhorov* distance between two finite Radon measures μ, ν on $\mathbb{R}^n$ is defined as

$$\eta_d(\mu,\nu) := \inf\left\{\varepsilon > 0 \,\middle|\, \forall B \subset \mathbb{R}^n \text{closed ball}, \begin{smallmatrix}\mu(B)\leq \nu(B^\varepsilon)+\varepsilon^d,\\ \nu(B)\leq u(B^\varepsilon)+\varepsilon^d\end{smallmatrix}\right\},$$

where $B^\varepsilon = \bigcup_{x\in B}\mathcal{B}(x,\varepsilon)$.

This second definition is straightforwardly shown to still be a distance. Now, to formulate their stability result, the authors introduce a combination of the previous distances on varifolds as

$$\delta(V,W) := \sup_{\substack{x\in\operatorname{supp}\|V\|\\ r>0}} \left\{\frac{\Delta_{\mathring{\mathcal{B}}(x,r)}(\|V\|,\|W\|)}{(\eta_d(\|V\|,\|W\|)+r)^d}\right\}. \tag{4}$$

Note that this is not a distance, and not symmetric: for $V = \delta_x$ and $W = \frac{3}{4}\delta_y$ with $x \neq y$, $\Delta_{\mathring{\mathcal{B}}(x,r)}(\|V\|,\|W\|) \in \left\{1, \frac{1}{4}\right\}$ and equality with 1 is achieved for small enough r, giving $\delta(V,W) = \frac{1}{\eta_d(\|V\|,\|W\|)^d}$ and similarly, one gets $\delta(W,V) = \frac{1}{4\eta_d(\|V\|,\|W\|)^d} \neq \delta(V,W)$. One can also see that it only accounts for the mass of the varifolds, i.e. it can be null without implying equality of the compared measures. Nevertheless, the authors show that it goes to 0 when its arguments get close for the weak-* topology, and obtain a stability for the extended approximate mean curvature defined in (3), which consists in the main result of the paper stated as follows.

THEOREM 3.5. *For V a d-regular varifold (for a constant C_0) of finite mass, $(V_i)_i$ a sequence of d-varifolds weak-* converging to V such that their masses are all compactly supported in $K \subset \mathbb{R}^n$, (x_i) a sequence of $\mathbb{R}^n$ converging to $x \in M$, (ε_i) a sequence of $(0,1)$ converging to 0 and such that $\|x - x_i\| + \eta_d(\|V\|,\|V_i\|) \leq 8(1 + (2C_0)^{\frac{1}{d}} + C_0^{\frac{2}{d}})$. Then,*

(i) $\delta(V,V_i) \to 0$,

(ii) $\left|H_{\varepsilon_i}^\Pi(x_i,V_i) - H_{\varepsilon_i}^\Pi(x,V)\right| = O\left(\frac{\delta(V,V_i)+\|x-x_i\|}{\varepsilon_i^2}\right)$.

The authors prove (i) by contradiction, using definitions 3.3 of the flat distance and (4) of δ as suprema to define sequences of balls and functionals, and applying a result of analysis to extract a uniformly converging sequence of lower bounded infinity norm, but of limit shown to be null. The proof of (ii) is made through heavy computations. While this result is quite general, it gives a rather coarse bound in the sense that it is not simple to get an upper bound on $\delta(V, V_i)$ from bounds on the flat distance or Prokhorov distance given the definition of δ as a quotient of both, and since δ does not satisfy the triangle inequality, it is difficult to manipulate this bound to obtain further results. Thus, the rate of convergence in the above bound is basically untractable.

For the case of interest when the limit varifold is smooth, the authors derive the following corollary from theorem 3.5 and proposition 3.1.

Corollary 3.6. *For $V = \pi_M^{-1} \sharp \mathcal{H}^d_{|M}$ where M is a C^3 compact d-manifold without boundary, $(V_i)_i$ a sequence of d-varifolds weak-* converging to V such that their masses are all compactly supported in $K \subset \mathbb{R}^n$, (x_i) a sequence of $\mathbb{R}^n$ converging to $x \in M$, (ε_i) a sequence of $(0,1)$ converging to 0 and such that $\|x - x_i\| + \eta_d(\|V\|, \|V_i\|) = o(\varepsilon_i)$, then*

$$\left| H^{\Pi}_{\varepsilon_i}(x_i, V_i) - H(x, V) \right| = O\left(\frac{\delta(V, V_i) + \|x - x_i\|}{\varepsilon_i^2} + \varepsilon_i \right),$$

And thus $H^{\Pi}_{\varepsilon_i}(x_i, V_i) \longrightarrow H(x, V)$ as soon as $\sqrt{\delta(V, V_i) + \|x - x_i\|} = o(\varepsilon_i)$.

A problem arising when trying to apply the above result is that the requirements on ε_i are also impossible to verify in practice, meaning that one gets no information on how to choose the parameter ε: the above result states that it should be "large enough" compared to the uncomputable quantities but should still get "close" to 0.

3.3 Mean curvature motion scheme

Under the hypothesis that the observated point cloud $(x_i)_{i=1}^N$ is close to a manifold M, the authors follow "standard" approaches to try and define a point cloud varifold $V = \sum_{i=1}^N m_i \delta_{x_i, P_i}$ to approximate $\pi_M^{-1} \sharp \mathcal{H}^d_{|M}$. Under motivations that are not entirely clear to the author of this report, the masses m_i are chosen through regularization of $\sum_i \delta_{x_i}$ and $\mathcal{H}^d_{|M}$ by λ_δ defined for a kernel λ and $\delta > 0$ similarly as in section 2.3, and approximation of the regularized Hausdorff measure by a first order approximation written $C_\lambda \delta^d$ with C_λ the volume weighted by λ of the unit ball in $\mathbb{R}^d$. This yields masses

$$m_i = \frac{C_\lambda \delta^d}{\sum_{j=1}^N \lambda\left(\frac{\|x_i - x_j\|}{\delta}\right)},$$

and a particular case considered by the authors is when λ is the indicator function $\mathbb{I}_{(-1,1)}$, in which case the mass becomes

$$m_i = \frac{\omega_d \delta^d}{k_\delta}, \ k_\delta := \left| \left\{ j, \|x_j - x_i\| < \delta \right\} \right| \tag{5}$$

with ω_d the volume of the unit ball in $\mathbb{R}^d$. The authors note that no covergence result seems to be known for such an estimator, meaning that the stability shown in corollary 3.6 does not apply. We further discuss this scheme and propose a simpler one (with some convergence property under a different point of view) in section 4. As for the directions P_i, they are computed in a more intuitive way by choosing a neighborhood of x_i, and applying (possibly regularized) Principal Component Analysis to the resulting subset of points, with target dimension d being an estimate of the intrinsic dimension of the unknown M. This allows one to have a consistent

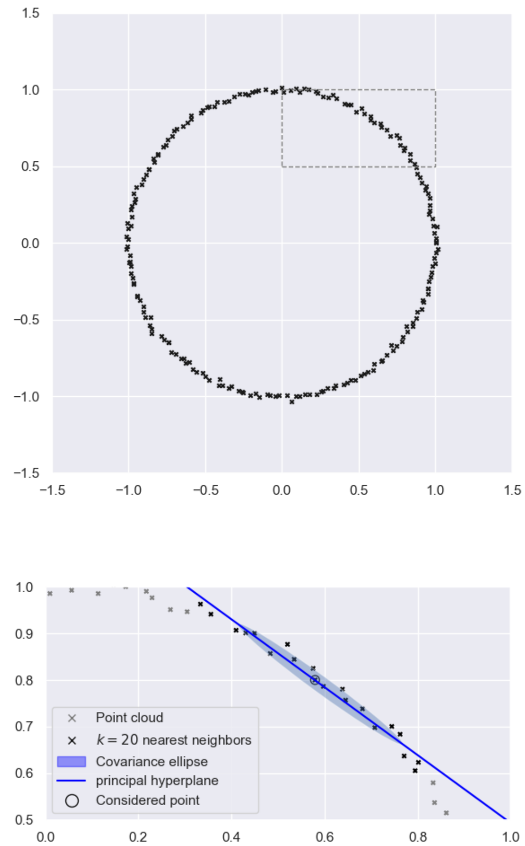

Figure 2: Computation of a local tangent P_i in the case of a noisy circle

local dimension robustly to noise. Such a process is illustrated in figure 2. With such estimators built, which for an input point cloud $X = (x_1, \ldots, x_N) \in \mathbb{R}^{n \times N}$ are denoted as $m_i(X)$ and $P_i(X)$, the authors formulate (time continuous) mean curvature motion in the case of point cloud varifolds. It writes as a system of differential equations with initial conditions: for given initial point cloud X_0, we wish to find $X(t) = (x_1(t), \ldots, x_N(t))$ such that

$$\begin{cases} \forall i, \ \frac{d}{dt} x_i(t) = H^{\Pi}_{\varepsilon}(x_i(t), V(t)), \\ X(0) = X_0, \end{cases} \tag{6}$$

where $V(t) := \sum_{i=1}^N m_i(X(t)) \delta_{x_i(t), P_i(X(t))}$. A solution of (6) is subsequently shown to have *planar barriers*, i.e. if a polytope contains the initial point cloud, then it contains the entirety of the trajectory of the flow. This only holds if locally, $\Pi(x_j(t) - x_i(t))$ points inside the convex hull of the point cloud, which always holds for $\Pi = 2\mathrm{Id}$, but is not always the case for the other choices given in proposition 3.1. The authors claim that for the choices $\Pi \in \left\{ -2\Pi_{P_j^\perp}, -2\Pi_{P_i^\perp} \circ \Pi_{P_j^\perp}, 2\Pi_{P_i^\perp} \right\}$ it appears to them that it could lead to a useful constraint on the definition of the P_i, without more elaboration, while for the remaining choices the assumptions do not hold for most reasonable choices of P_i. Considering having such a planar barrier property is desirable to avoid divergence of the flow, it seems that the work made in proposition 3.1 to generalize approximate mean curvature to more projectors is of limited use, as already 2 of the 6 proposed projectors fail to obtain such property, and in fact only one other than 2Id will be kept in light of numerical experiments. Similarly, the authors show a sphere comparison principle, stating that the flow is contained in a sphere of radius varying with t, but said radius is only shown to decrease for $\Pi = 2\mathrm{Id}$ and may be increasing for other choices of projector depending on the way the P_i are computed.

Nevertheless, the authors go on to derive the time discretization of 6 with step τ, as the following semi-implicit scheme with the obvious adapted notations:

$$x_i^{k+1} = x_i^k + \tau H_\varepsilon^\Pi \left(x_i^{k+1}, \sum_{i=1}^N m_i(X_k) \delta_{x_i^{k+1}, P_i(X_k)} \right). \qquad (7)$$

It is semi-implicit in the sense that it is implicit with respect to the positions x_i^k but explicit in the masses and directions. It is then shown that (7) amounts to solving an invertible linear system, allowing one to compute the flow although with relatively high computational cost due to the complexity of inverting a linear system of large size over possibly many iterations for small τ. The authors then derive similar plane barrier and sphere comparison properties as before in this time-discrete setting.

3.4 Numerical experiments

The paper presents some numerical results, for which we quickly see the irrelevance of most of the projectors which make the motion diverge or be highly sensitive to noise. Nevertheless, the case $\Pi = \Pi_{P_i^\perp}$ fares decently against toy cases, even when triple points emerge. The authors do not always compare these results with the base case $\Pi = 2\mathrm{Id}$ though, and do not show any other algorithms for that matter. It is therefore difficult to make clear conclusions about the effectiveness of the method in practice, given that no complex, real life datasets are investigated.

4 EXTENSIONS

We explicit a few directions in which one could attempt to extend the framework developed in the paper. First, consider definition 2.2 of a smooth varifold. As seen in section 2, the choice of the Hausdorff measure can be seen as the uniform distribution on the considered manifold. Thus, we propose to introduce a generalized smooth varifold as follows.

Definition 4.1. A *generalized smooth varifold* V associated to a d-manifold M is such that

$$V = \pi_M^{-1} \sharp \|V\|.$$

Such a generalization could be useful in the sense that, in a statistical setting, the observations may be sampled non-uniformly over the surface. Considering the application to laser scans of surfaces mentioned by the authors, some nontrivial yet realistic surfaces may have locations harder to reach for the light beams, meaning they will be sampled less often that the rest of the shape. A simple example is provided figure 3.

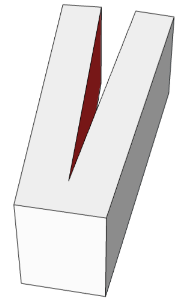

Figure 3: V-shaped surface on which the inside (shaded red) is less likely to be sampled from a laser scan.

Thus, one could attempt to study the behavior of such varifolds to get a more general estimator of the mass that could be more robust to non-uniform samplings of the surface of interest. Secondly, in (5), one can remark that if the points (x_i) are seen as an i.i.d. uniform sampling of M, then the above coefficient would intuitively behave like $\frac{1}{n}$ as the number of observations n grows, since $\frac{k_\delta}{n}$ can be seen as Monte Carlo estimation of the volume of the d-ball of radius δ on M, which is given by the numerator. The simpler scheme of setting $m_i = \frac{1}{n}$ could have been suggested from the start: it is interpretable as an empirical process, which has the following convergence property in a quite general case.

Proposition 4.2. *Let V be a varifold of finite mass, and $X_1, \ldots, X_N$ be i.i.d random variables valued in $\mathbb{R}^n$ and of law $\frac{\|V\|}{\|V\|(\mathbb{R}^n)}$. Then, with probability 1,*

$$\frac{1}{N} \sum_{i=1}^N \delta_{X_i} \rightharpoonup^* \frac{\|V\|}{\|V\|(\mathbb{R}^n)}$$

where $\rightharpoonup^*$ denotes the weak-* convergence.

In particular, since the mean curvature is invariant to rescalings of the mass of a varifold, and since theorem 3.5 in fact only relies on weak-* convergence of the mass, it could be applied for large N to recover some convergence of approximate mean curvature. The above proposition is straightforward to prove by applying the definition of weak-* convergence and using the law of large numbers. Similarly, one could study the convergence in this statistical point of view of the previously presented estimates of the directions.

5 CONCLUSION

The work proposed in [7] has for main contributions the generalization of approximate mean curvature previously explored in [6] along with convergence bounds, and the derivation of a time-discretization of the (approximate) mean curvature flow in a semi-implicit scheme. We highlighted limitations in the assumption of uniform sampling, the lack of motivation and properties of the newly introduced projectors, as well as the impracticality of the shown convergence bounds. Overall, the presented methods seem to lack clear purpose i.e. a clear problem to solve, and is presented without much intuitivity (the intuitions given in this paper are original and were absent in [7]), which contributes to the barrier of entry to the field of varifold theory. While the numerical experiments seem to show some merit in favor of the authors' method, it is not compared to other existing procedures and its relevance is thus difficult to assess. This paper seems to highlight both some strengths of the varifold framework in its generalty, and its main weaknesses in its technicality. On a final note, we present some slight extensions of the paper that could hope to be developed and fill in some of the found gaps, namely the non-uniform smooth varifold setting as well as a simpler, theoretically supported estimator for the masses of a discrete approximation of a smooth varifold.

REFERENCES

[1] William K. Allard. 1972. On the First Variation of a Varifold. *Annals of Mathematics* 95, 3 (1972), 417–491. http://www.jstor.org/stable/1970868

[2] Frederic J. Almgren. 1965. The theory of Varifolds.

[3] K. A. Brakke. 1978. *The Motion of a Surface by Its Mean Curvature.* Number 20 in Mathematical Notes. Princeton University Press.

[4] Blanche Buet. 2014. *Discrete varifolds and surface approximation: representation, curvature, rectifiability.* Ph. D. Dissertation. Universite Claude Bernard Lyon 1.

[5] Blanche Buet. 2015. Quantitative conditions of rectifiability for varifolds. *Annales de l'Institut Fourier* 65, 6 (2015), 2449–2506. https://doi.org/10.5802/aif.2993

[6] Blanche Buet, Gian Leonardi, and Simon Masnou. 2017. A Varifold Approach to Surface Approximation. *Archive for Rational Mechanics and Analysis* 226 (11 2017). https://doi.org/10.1007/s00205-017-1141-0

[7] Buet, Blanche and Rumpf, Martin. 2022. Mean curvature motion of point cloud varifolds. *ESAIM: M2AN* 56, 5 (2022), 1773–1808. https://doi.org/10.1051/m2an/2022047

[8] Nicolas Charon and Alain Trouvé. 2013. The Varifold Representation of Nonoriented Shapes for Diffeomorphic Registration. *SIAM Journal on Imaging Sciences* 6, 4 (Jan. 2013), 2547–2580. https://doi.org/10.1137/130918885

[9] Mathieu Desbrun, Mark Meyer, Peter Schröder, and Alan H. Barr. 1999. Implicit Fairing of Irregular Meshes Using Diffusion and Curvature Flow. In *Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH '99).* ACM Press/Addison-Wesley Publishing Co., USA, 317–324. https://doi.org/10.1145/311535.311576

[10] Lawrence C. Evans and Joel Spruck. 1995. Motion of Level Sets by Mean Curvature IV. *The Journal of Geometric Analysis* 5, 1 (01 03 1995), 77–114. https://doi.org/10.1007/BF02926443

[11] Irène Kaltenmark, Benjamin Charlier, and Nicolas Charon. 2017. A General Framework for Curve and Surface Comparison and Registration with Oriented Varifolds. In *2017 IEEE Conference on Computer Vision and Pattern Recognition (CVPR).* 4580–4589. https://doi.org/10.1109/CVPR.2017.487

[12] Laurent Chisholm Young. 1951. Surfaces paramétriques généralisées. *Bulletin de la Société Mathématique de France* 79 (1951), 59–84. https://doi.org/10.24033/bsmf.1419