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FibeRed: Fiberwise Dimensionality Reduction of Topologically Complex Data with Vector Bundles

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Plan

1. Background on vector bundles (Chen)

2. Main contributions (Mathis H)

3. Numerical experiments, criticism (Mathis R)

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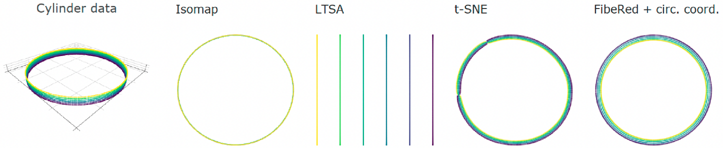
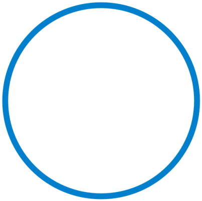
Motivation and Utility



- Aims:preserve the large-scale topology of the data while reducing the dimensionality of the local geometric features.
- How to do this? Find a model takes account to both topological and local geometry.
- Our model is : Vector bundle

Vector bundle

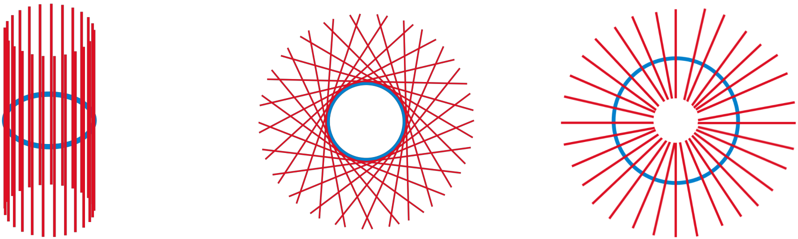


(A) Base space (manifold)

(B) Fiber (vector space)

Vector bundle $=$ Every point in base space "grows" a vector space(fiber)

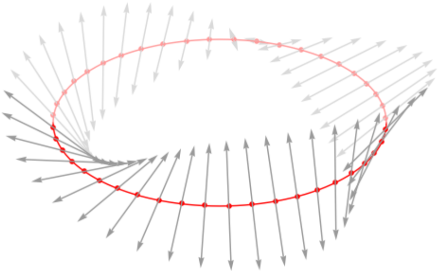
Vector bundles



(A) $= S^1 \times \mathbb{R}^1$ (B) $\sim S^1 \times \mathbb{R}^1$ (C) $\sim S^1 \times \mathbb{R}^1$

FIGURE 2: Three trivial vector bundles with same base space

Non trivial Vector bundle

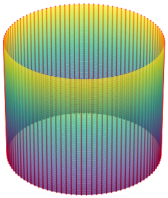


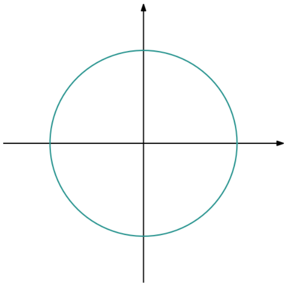
Bundle with Möbius strip topology

$\not\approx S^1 \times \mathbb{R}^1$

But can be embedded into $S^1 \times \mathbb{R}^2$

Circular Coordinates





$$\pi : \quad \mathcal{X} \longrightarrow S^1$$

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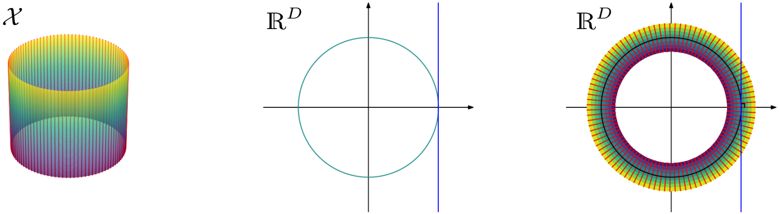
Problem formulation

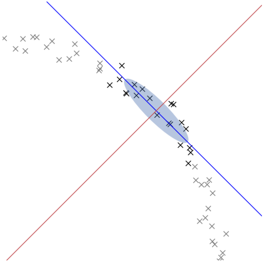
Problem

Given an initial bundle $\pi : \mathcal{X} \to \mathbb{R}^D$ capturing the global topology of $\mathcal{X}$, refine it into a map $\tilde{\pi} : \mathcal{X} \to \mathbb{R}^D$ which additionally takes into account the local geometry.



Algorithm: FibeRed

- Input: π, $X \subset \mathcal{X}$, distance matrix on X, estimate intrinsic dimensions e of $\mathcal{B} \coloneqq \pi(\mathcal{X})$ and d of $\mathcal{X}$. $B \coloneqq \pi(X)$.
- Build charts:
  - Compute a cover $(U_i)_i$ of B and its nerve, inducing a cover $(X_i)_i \coloneqq (\pi^{-1}(U_i))_i$ of X.
  - Deduce local coordinates on X, tangent and normal coordinates on B with linear dimensionality reduction.



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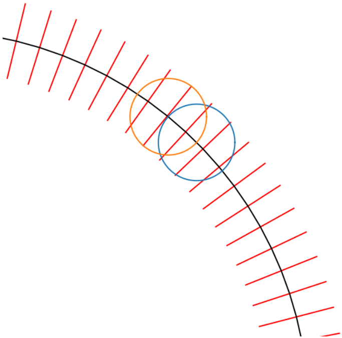
Algorithm: FibeRed

- Estimate the reach from the cover
- Preserve large scale topology:
  - Estimate cocycles for the bundle π and the normal bundle, thus refining the charts to be more consistent on edges of the nerve.



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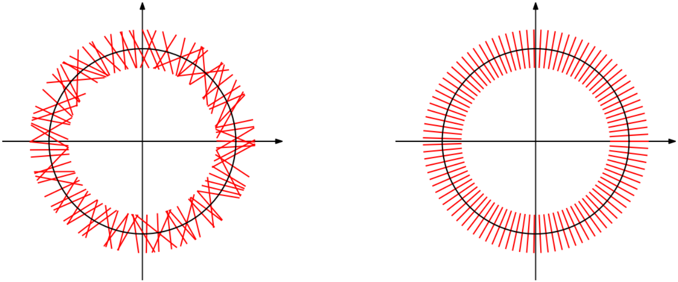
Algorithm: FibeRed

- Refine the embedding to be more faithful to local geometry:
  - Align the fibers of π to be as close as possible to the normal fibers
- Combine the obtained elements to get the final map



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Plan

Authors experiment

Inputs :

- Points
- Distance matrix
- Initial map

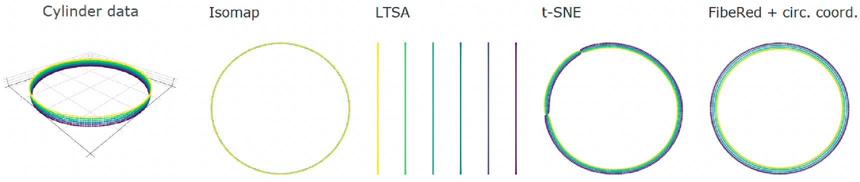
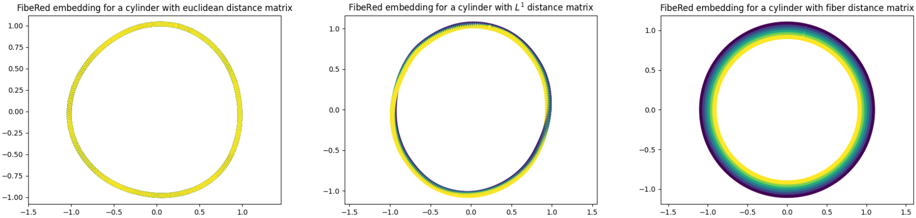


FIGURE 3: Cylinder experiment

Varying distance



(A) Euclidean distance (B) L_1 distance (C) Geodesic distance

Figure 4: Output embedding of a ring for 3 distances

Introducing noise

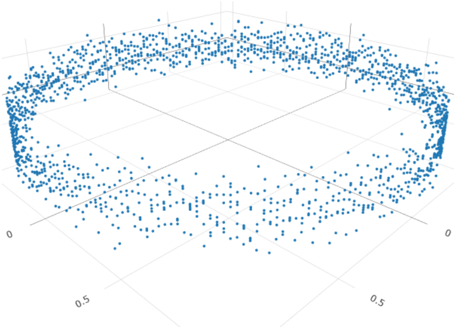
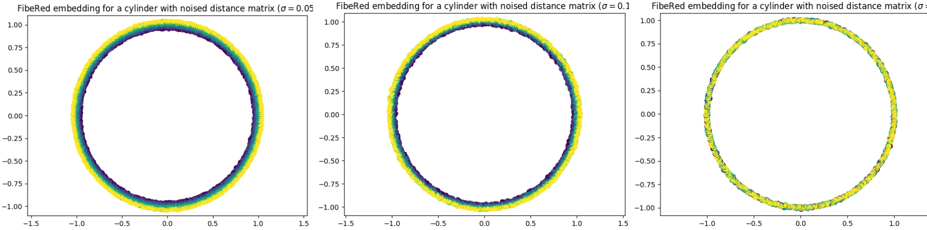


Figure 5: Noised cylinder

Robustness to noise



(A) $\sigma = 0.05$ (B) $\sigma = 0.1$ (C) $\sigma = 0.5$

FIGURE 6: Impact of increasingly intense noise on the resulting embedding

Concluding remarks & criticism

- No results on real-life data
- Heavy preprocessing

Appendix

Formal problem

Problem

Given an embedding $\iota : \mathcal{B} \to \mathbb{R}^D$, find a fiberwise isometric embedding $\bar{\iota} : \mathcal{X} \to \mathbb{R}^D$ that extends ι in the sense that $\bar{\iota} \circ s_0 = \iota$, and that is orthogonal to $\mathcal{B}$, in the sense that $\bar{\iota}(\pi^{-1}(b)) \perp \iota(T_b B)$ for all $b \in \mathcal{B}$.

Property

The above admits a solution if and only if there exists a morphism $\mathcal{X} \to N$ of vector bundles over $\mathcal{B}$ that is an isometry in each fiber. This is also equivalent to the existence of maps $\Phi = \{\Phi_i : U_i \to \bigvee(r, D-e)\}$ such that

$$\Phi_i(b)\Omega_{ij}(b) = \Theta_{ij}(b)\Phi_j(b), Z \text{ for all } i \text{ and } j \text{ and } b \in U_i \cap U_j. \quad (1)$$

Where $\Omega = \{\Omega_{ij} : U_i \cap U_j \to O(r)\}$ is a cocycle with associated vector bundle π defined as the unique set of maps satisfying

$$\Omega_{ij}(\pi(x))\, f_j(x) = f_i(x), \text{ for all } x \in \mathcal{X}_i \cap \mathcal{X}_j, \quad (2)$$

Representation formula

Property

The following formula gives an embedding $\mathrm{disk}(\pi) \to \mathbb{R}^D$:

$$x \;\mapsto\; c\,\tau \cdot \alpha_i(\pi(x))\,\Phi_i(\pi(x))\,f_i(x) + \iota(\pi(x)), \quad \text{for } \pi(x) \in U_i \tag{3}$$

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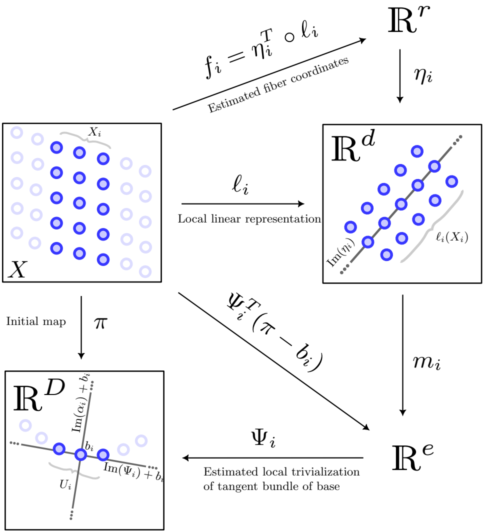
Main constructions



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