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Mean Curvature Motion of Point Cloud Varifolds

Mathis Hardion

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Context

- Varifold framework as a general model for surfaces
  - Relatively scarcely used
  - Authors aim to show its relevance in computational geometry
- Mean curvature flow
  - Surface fairing
- Approximate mean curvature

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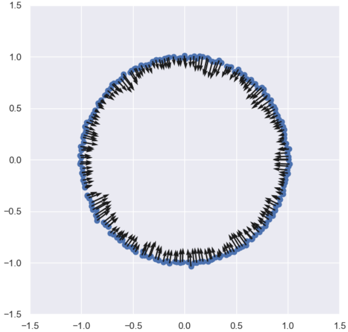
Main contributions

- Slight generalization of approximate mean curvature by considering 5 additional projectors instead of the identity
  - Not strongly motivated
  - Most choices of projector end up discarded
- Extension of convergence properties
  - Heavy, computational proofs
  - Uninformative bounds, hypotheses next to impossible to check in practice

Main contributions

- Derivation of corresponding (approximate) mean curvature flow for point cloud varifolds
  - ”Standard” estimation of masses and tangent planes
  - Semi-implicit scheme: more accurate, but computationally costly for large point clouds
  - Desirable properties (planar barrier, sphere comparison) are shown in specific cases, which are always verified only in the case of the identity projector



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1. Context
2. Main contributions, limitations
3. Extensions
4. Conclusion

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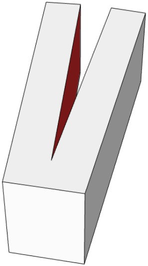
Non-uniform smooth varifolds



Definition

A *generalized smooth varifold* V associated to a d-manifold M is such that

$$V = \pi_M^{-1} \sharp \|V\| = \|V\| \left(\{ x \in M, (x, T_x M) \in \cdot \} \right) .$$

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A simple mass estimator

Property

Let V be a varifold of finite mass, and $X_1, \ldots, X_N$ be i.i.d random variables valued in $\mathbb{R}^n$ and of law $\frac{\|V\|}{\|V\|(\mathbb{R}^n)}$. Then, with probability 1,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{X_i} \rightharpoonup^* \frac{\|V\|}{\|V\|(\mathbb{R}^n)}$$

where $\rightharpoonup^*$ denotes the weak-* convergence.

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Concluding remarks

- Illustrates the weaknesses of the varifold framework and some of its strengths
- Lack of a clear problem to solve
- Illustration only on toy data, no comparison with other algorithms

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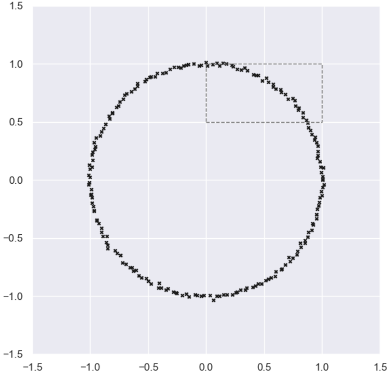
Mass estimation

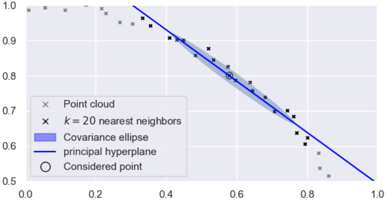
$$m_i = \frac{C_\lambda \delta^d}{\sum_{j=1}^N \lambda\left(\frac{\|x_i - x_j\|}{\delta}\right)},$$

$\lambda = \mathbb{I}_{(-1,1)}$:

$$m_i = \frac{\omega_d \delta^d}{k_\delta},\ k_\delta := |\{j, \|x_j - x_i\| < \delta\}|$$

Tangent space estimation





Main results

Property

For $d = n-1$, $M \subset \mathbb{R}^n$ a $\mathscr{C}^2$ d-manifold of mean curvature vector $H : M \to \mathbb{R}^n$, $V = \pi_M^{-1} \sharp \mathscr{H}^d_{|M}$, and for $\Pi \in \left\{ \Pi_P, -2\Pi_{P^\perp}, 2\mathrm{Id}, \Pi_{(T_x M)^\perp} \circ \Pi_P, -2\Pi_{(T_x M)^\perp} \circ \Pi_{P^\perp}, 2\Pi_{(T_x M)^\perp} \right\}$, then for any $x \in M$,

$$H_\varepsilon^\Pi (x, V) \xrightarrow[\varepsilon \to 0]{} H(x).$$

Additionally, if M is of class $\mathscr{C}^3$, $\left| H_\varepsilon^\Pi(x, V) - H(x) \right| = O(\varepsilon)$.

Definition

$$\delta(V, W) := \sup_{\substack{x \in \operatorname{supp} \|V\| \\ r > 0}} \left\{ \frac{\Delta_{\mathring{\mathscr{B}}(x,r)}(\|V\|, \|W\|)}{(\eta_d(\|V\|, \|W\|) + r)^d} \right\}.$$

Main results

Theorem: F

or V a d-regular varifold (for a constant C_0) of finite mass, $(V_i)_i$ a sequence of d-varifolds weak-* converging to V such that their masses are all compactly supported in $K \subset \mathbb{R}^n$, (x_i) a sequence of $\mathbb{R}^n$ converging to $x \in M$, (ε_i) a sequence of $(0,1)$ converging to 0 and such that $\|x - x_i\| + \eta_d(\|V\|, \|V_i\|) \leq 8\left(1 + (2C_0)^{\frac{1}{d}} + C_0^{\frac{2}{d}}\right)$. Then,

(i) $\delta(V, V_i) \to 0$,

(ii) $\left|H^{\Pi}_{\varepsilon_i}(x_i, V_i) - H^{\Pi}_{\varepsilon_i}(x, V)\right| = O\left(\frac{\delta(V,V_i)+\|x-x_i\|}{\varepsilon_i^2}\right)$.

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Main results

Corollary

For $V = \pi_M^{-1}\sharp\mathscr{H}^d_{|M}$ where M is a $\mathscr{C}^3$ compact d-manifold without boundary, $(V_i)_i$ a sequence of d-varifolds weak-* converging to V such that their masses are all compactly supported in $K \subset \mathbb{R}^n$, (x_i) a sequence of $\mathbb{R}^n$ converging to $x \in M$, (ε_i) a sequence of $(0,1)$ converging to 0 and such that $\|x - x_i\| + \eta_d(\|V\|, \|V_i\|) = o(\varepsilon_i)$, then

$$\left| H^{\Pi}_{\varepsilon_i}(x_i, V_i) - H(x, V) \right| = O\left(\frac{\delta(V, V_i) + \|x - x_i\|}{\varepsilon_i^2} + \varepsilon_i \right),$$

And thus $H^{\Pi}_{\varepsilon_i}(x_i, V_i) \to H(x, V)$ as soon as $\sqrt{\delta(V, V_i) + \|x - x_i\|} = o(\varepsilon_i)$.

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